THE USE OF POINTWISE INTERPOLATION IN DOMAIN DECOMPOSITION METHODS WITH NON-NESTED MESHES *

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Abstract. In this paper, we develop a new technique, and a corresponding theory, for Schwarz type overlapping domain decomposition methods for solving large sparse linear systems which arise from finite element discretization of elliptic partial differential equations. The theory provides an optimal convergence of an additive Schwarz algorithm that is constructed with a non-nested coarse space, and a not necessarily shape regular subdomain partitioning. The theory is also applicable to the graph partitioning algorithms recently developed, [5, 15], for problems defined on unstructured meshes.

Key words. unstructured grids, non-nested meshes, domain decomposition, iterative method.

AMS(MOS) subject classifications. 65F10, 65N30.

1. Introduction. Considerable interest has developed in Schwarz type overlapping domain decomposition methods for the numerical solution of partial differential equations; see for examples [2, 3, 4, 6, 7, 12, 13, 18] and the references therein. This class of methods offers a great deal of parallelism and is very promising for modern parallel computers. The success of the methods depends heavily on the existence of a uniformly, or nearly uniformly, bounded decomposition of the function space in which the problem is defined. In this paper, we further enrich the Schwarz theory by providing a new technique of constructing a uniformly bounded decomposition of the problem space, which is more flexible and convenient for large, geometrically complicated, practical problems. It has a convergence rate that is similar to that of the regular Dryja-Widlund type decomposition [13], and does not require the coarse space to be a subspace of the original finite element space, in which the partial differential equation is discretized. Nor does it require that the collection of the un-extended subdomains forms a regular finite element subdivision.

We shall only discuss a two-level additive Schwarz algorithm, with a coarse and a fine grid. It is well known that the fine grid determines the accuracy of the discrete problem and that the only role of the coarse grid is to accelerate the convergence of the iterative method. In this paper, we try to minimize the inter-connection between the two grids by using a not necessarily nested coarse grid. As a result, the same coarse grid can be used even if the fine grid is locally refined, or re-meshed, to deal with the local singularity of the underlining problem. There are a number of ways to handle the communication between the two grids, in this paper, we insist on the computationally simplest one, i.e., pointwise interpolation. We show that this, sometimes troublesome, interpolation operator behaves well in both two- and three-dimensional space in our applications. For technical reasons, we assume that the coarse mesh is quasi-uniform, however, no such assumption is needed for the fine mesh. Other recent developments along this line can be found in [9, 10, 11, 19, 20]. Some theory and experiments with the pointwise interpolation in the context of non-nested multigrid methods can be found in [1, 8, 16, 17], and references therein.

In [14], Dryja and Widlund developed a general theory for Schwarz type algorithms which has a convergence rate characterized by the quantity $(1 + H/\delta)$, where H measures

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the diameters of the subdomains (as well as the coarse mesh size) and δ the overlap between neighboring subdomains. This quantity indicates that subdomains with uniform aspect ratio is desired. In this paper, we develop a result involving $\min\{1 + H_c^2/\delta^2,\ 1 + H/\delta + H_c/\delta \cdot H_c/H\}$, where H_c is the size of the coarse grid, which generally has nothing to do with the subdomain diameter H. The first quantity is independent of H. This allows us to use subdomains of arbitrary shape. As a consequence, our theory applies to the type of unstructured mesh problems decomposed by some graph-based partitioning techniques discussed by Cai and Saad in [5]. The second quantity reduces to that of Dryja and Widlund when $H \approx H_c$, and comes into play in the case of small overlap.

When solving a system of equations arising from the discretization of non-selfadjoint, or indefinite, or nonlinear elliptic problems by a Schwarz type algorithm, a fine enough coarse mesh space is usually used in order to make the convergence rate optimal, see e.g., [6, 7]. In such a case, using the Dryja-Widlund construction [13] would normally result in a large number of subdomains that have to be combined in order for the number of subproblems to fit the number of processors of a parallel computer. With our new construction, the size of the coarse mesh is totally independent of the number of subdomains.

In this paper, we shall focus only on a simple self-adjoint model problem, namely the homogeneous Dirichlet boundary value problem: Find $u_h^* \in V_h \subset H_0^1(\Omega)$ such that

$$a(u_h^*, v) = (f, v), \qquad \forall v \in V_h, \tag{1}$$

where the bilinear form $a(u, v)$ is defined by $a(u, v) = \int_\Omega \nabla u \nabla v dx$, $f(x) \in L^2(\Omega)$ is given, and Ω is an open bounded polygon in R^d ($d = 2$ or 3), with boundary $\partial\Omega$. We shall use $a(\cdot, \cdot)$ and $\|\cdot\|_a$ to denote the inner product and norm of $H_0^1(\Omega)$. To introduce the finite element discretization and the finite element space V_h, we let $\Omega_h = \{k_i\}$ be a standard finite element triangulation of Ω that satisfies the minimal angle condition, i.e., in two dimensional case $\gamma_k \geq \gamma_0 > 0$, for any $k \in \Omega_h$. Here γ_k is the minimal interior angle of $k \in \Omega_h$ and γ_0 a constant. We do not assume that the triangulation is quasi-uniform. We allow the use of highly refined unstructured meshes. We define the corresponding finite element space $V_h \subset H_0^1(\Omega)$ as the regular piecewise linear continuous triangular finite element space on Ω. Let us denote by $\bar{h}$ the maximum diameter of this finite element mesh which will be used later to restrict the size of the coarse grid.

Throughout this paper, c and C, with or without subscripts, denote generic, strictly positive constants which are independent of any of the mesh parameters.

2. A non-nested coarse mesh space. We begin by introducing several notations. Let $\Omega_{H_c} = \{\tau_i\}$ be a quasi-uniform triangulation of Ω and τ_i one of the triangles whose diameter is of order H_c. Ω_{H_c} will be referred to as the coarse grid. Here H_c is the maximum diameter of this coarse triangulation. We assume, throughout this paper, that each fine triangle intersects with a finite number of coarse triangles,

$$\bar{h} \leq C\, H_c, \text{ and} \tag{2}$$

$$|k_i| \leq C\, |\tau_j|, \quad \text{if } k_i \cap \tau_j \neq \emptyset. \tag{3}$$

Here and in the rest of the paper, $|\cdot|$ means the area in R^2 and the volume in R^3. Let $V_{H_c} \subset H_0^1(\Omega)$ be a shape-regular finite element space over Ω consisting of piecewise linear continuous functions. Note that, in general, $V_{H_c} \not\subset V_h$, and it is not necessary for V_{H_c} to have the same type of elements as V_h. Let $\Pi_h : C^0(\bar{\Omega}) \to V_h$ be the usual piecewise linear continuous interpolation operator, which uses values only at the nodal points of the fine mesh triangulation. This operator has the following properties.

Lemma 2.1. *There exists a constant $C > 0$, independent of $\bar{h}$, H_c, such that*

(i) $\|\Pi_h v\|_a \leq C\|v\|_a, \quad \forall v \in V_{H_c}$;

(ii) $\|v - \Pi_h v\|_{L^2(\Omega)} \leq C\bar{h}\|v\|_a, \quad \forall v \in V_{H_c}$.

We note that estimates (i) and (ii) do not hold if v is an arbitrary function in $H^1(\Omega)$. However, we need the bounds only for functions in the subspace V_{H_c}. A proof of Lemma 2.1 will be given in §4. Let

$$V_0 \equiv \Pi_h V_{H_c} \equiv \{v \in V_h, \text{ there exists } w \in V_{H_c}, \text{ such that } v = \Pi_h w\},$$

which is a subspace of V_h. We shall use the L^2 projection operator $Q_{H_c} : H_0^1(\Omega) \to V_{H_c}$ defined by $(Q_{H_c} u, v) = (u, v)$, for any $u \in H_0^1(\Omega)$, and $v \in V_{H_c}$.

We now partition Ω into non-overlapping subdomains $\{\Omega_i\}$, such that no $\partial\Omega_i$ cuts through any elements k_i, and $\bar{\Omega} = \cup_{i=1}^N \bar{\Omega}_i$. Note that we do not assume that $\{\Omega_i\}$ forms a regular finite element subdivision of Ω, nor that the diameters of Ω_i are of the same order. In practice, a graph based partitioning technique, such as those introduced in [5, 15], can often be used to obtain Ω_i, especially if Ω_h is an unstructured grid. To obtain an overlapping decomposition of Ω, we extend each Ω_i to a larger subdomain $\Omega_i' \supset \Omega_i$, which is also assumed not to cut any fine mesh triangles, such that distance$(\partial\Omega_i' \cap \Omega, \partial\Omega_i \cap \Omega) \geq c\delta, \quad \forall i$, for a constant $c > 0$. Here $\delta > 0$ will be referred to as the overlapping size. For each Ω_i', we define a finite element space $V_i \equiv V_h \cap H_0^1(\Omega_i')$ and extended by zero outside Ω_i'. In the next lemma, we prove that the decomposition $V_h = V_0 + V_1 + \cdots + V_N$ exists, and is uniformly bounded.

Lemma 2.2. *For any $v \in V_h$, there exist $v_i \in V_i$, $i = 1, \cdots, N$, and $v_0' \in V_{H_c}$ such that $v = \Pi_h v_0' + v_1 + \cdots + v_N$, and in addition, there exists a constant $C_0 > 0$ independent of the mesh parameters, such that*

$$\|v_0'\|_a^2 + \sum_{i=1}^N \|v_i\|_a^2 \leq C_0 \left(1 + \frac{H_c^2}{\delta^2}\right) \|v\|_a^2, \quad \forall v \in V_h. \tag{4}$$

Proof. For any $v \in V_h$, let $v_0' = Q_{H_c} v \in V_{H_c}$, $v_0 = \Pi_h v_0' \in V_0$ and $w = v - v_0 \in V_h$. Because of the boundedness of Q_{H_c} in the H_0^1 norm, we clearly have $\|v_0'\|_a \leq C\|v\|_a$. Let $\{\theta_i(x)\}$ be a partition of unity of Ω corresponding to $\{\Omega_i'\}$, such that $|\nabla\theta_i|_2 \leq C/\delta$ and $\sum_{i=1}^N \theta_i(x) \equiv 1$ ($|\cdot|_2$ is the usual Euclidean norm in R^2 or R^3). Of course, θ_i are smooth and $0 \leq \theta_i \leq 1$. We define $v_i = \Pi_h(\theta_i w) \in V_i$. It is easy to see that $v_0 + \sum_{i=1}^N v_i = v$, therefore the existence of such a decomposition is proved.

Let k be a single triangular element in Ω_i' with diameter h. We assume that the average of θ_i over k is $\bar{\theta}_{i,k}$. It can be seen that

$$|v_i|_{H^1(k)}^2 \leq 2|\bar{\theta}_{i,k}\Pi_h w|_{H^1(k)}^2 + 2|\Pi_h\left((\theta_i - \bar{\theta}_{i,k})w\right)|_{H^1(k)}^2. \tag{5}$$

Because $\Pi_h w = w$, the first term on the right-hand side of this inequality presents no problems. We next estimate the second term. With the help of the element-wise inverse inequality, we have

$$\begin{aligned} |\Pi_h\left((\theta_i - \bar{\theta}_{i,k})w\right)|_{H^1(k)} &\leq C\,\frac{1}{h}\,\|\Pi_h\left((\theta_i - \bar{\theta}_{i,k})w\right)\|_{L^2(k)} \\ &\leq C\,\frac{1}{h}\,\frac{h}{\delta}\,\|w\|_{L^2(k)} \;=\; C\,\frac{1}{\delta}\,\|w\|_{L^2(k)}, \end{aligned} \tag{6}$$

where C depends only on the finite element subdivision of Ω. The fact $|\theta_i - \bar{\theta}_{i,k}| \leq C\, h/\delta$ is also used. By taking the sum over all elements $k \in \Omega_i^{'}$, we arrive at the estimate

$$|v_i|^2_{H^1(\Omega_i')} \leq C \left(|w|^2_{H^1(\Omega_i')} + \frac{1}{\delta^2} \|w\|^2_{L^2(\Omega_i')} \right), \tag{7}$$

which implies that

$$\sum_{i=1}^{N} |v_i|^2_{H^1(\Omega_i')} \leq C \left(\|w\|^2_a + \frac{1}{\delta^2} \|w\|^2_{L^2(\Omega)} \right). \tag{8}$$

Here the fact that each point in Ω is covered by only a finite number of overlapping subdomains is assumed. To bound the first term on the right-hand side of (8) in terms of $\|v\|_a$, we use the boundedness of the operators Π_h and Q_{H_c} in the H_0^1 norm, i.e.,

$$\|w\|_a = \|v - \Pi_h Q_{H_c} v\|_a \leq C\,(\|v\|_a + \|Q_{H_c} v\|_a) \leq C \|v\|_a. \tag{9}$$

To estimate the second term on the right-hand side of (8) in terms of $\|v\|_a$, we need the L^2 regularity estimates of Π_h((ii) of Lemma 2.1) and Q_{H_c}, which give us

$$\|w\|_{L^2(\Omega)} \leq \|v - Q_{H_c} v\|_{L^2(\Omega)} + \|Q_{H_c} v - \Pi_h Q_{H_c} v\|_{L^2(\Omega)} \leq C H_c \|v\|_a. \tag{10}$$

Recall that $\bar{h} \leq C\, H_c$ by assumption. The proof of the lemma thus follows immediately by combining the estimates (8), (9) and (10). □

3. An estimate for the small overlap case. The decomposition bound (4), provided in the previous section, grows at a rate proportional to $1/\delta^2$, which is rather large when a small overlap is used. In this section, we discuss an alternative estimate, for the same decomposition described in Lemma 2.2, and prove that it is in fact proportional only to $1/\delta$. However, for the small overlap case, we do need to assume that the $\Omega_i^{'}$s have approximately the same size, i.e., if H_i is the diameter of $\Omega_i^{'}$, then there exists a constant β, such that $1 \geq \min\{H_i\}/\max\{H_i\} \geq \beta$.

LEMMA 3.1 (DRYJA AND WIDLUND[14]). *Let $\Gamma_{\delta,i} \subset \Omega_i^{'}$ be the set of points that is within a distance of δ of $\partial\Omega_i \cap \Omega$. Then*

$$\|w\|^2_{L^2(\Gamma_{\delta,i})} \leq C\delta^2 \left((1 + \frac{H}{\delta}) |w|^2_{H^1(\Omega_i')} + \frac{1}{\delta H} \|w\|^2_{L^2(\Omega_i')} \right), \quad \forall w \in H^1(\Omega_i^{'}), \;\; \forall i. \tag{11}$$

Here H is the maximum diameter of these $\Omega_i^{'}$s. The constant C may depend on β.

LEMMA 3.2. *The same decomposition described in Lemma 2.2 exists and is bounded in the sense that there exists a constant $C_0 > 0$ independent of the mesh parameters, such that*

$$\|v_0^{'}\|^2_a + \sum_{i=1}^{N} \|v_i\|^2_a \leq C_0 \left(1 + \frac{H}{\delta} + \frac{H_c^2}{\delta H} \right) \|v\|^2_a, \quad \forall v \in V_h. \tag{12}$$

Proof. The proof is nearly the same as that for Lemma 2.2, except that we make use of the fact that $\theta_i - \bar{\theta}_{i,k} = 0$ if $k \subset \Omega_i^{'} \setminus \Gamma_{\delta,i}$. □

We note that the aspect ratio, or diameter, of the subdomains, H, appears in the estimate in the case when small overlap is being used. The factor H is introduced into the estimate by Lemma 3.1. We do not know whether it can be removed, or replaced by a quantity that is independent of the subdomain aspect ratio. We also comment that if $H_c \approx H$, then the result of Lemma 3.2 coincides with that of [14].

4. The boundedness of Π_h. If the interpolation operator Π_h is considered as a map from the space $H_0^1(\Omega)$ to V_h, then it generally does not satisfy the bounds stated in Lemma 2.1, because the values of a $H^1(\Omega)$-function are not necessarily well defined at the mesh points. In this section, we prove, however, that Π_h, restricted to the subspace $V_{H_c} \subset H_0^1(\Omega)$, is indeed bounded. In a trivial case, when $V_{H_c} \subset V_h$ then (i) of Lemma 2.1 holds as an equality with $C = 1$, and (ii) holds with $C = 0$. The elementary proof provided in this section applies to both $d = 2$ and 3. Several different proofs have been obtained recently, see [9, 11, 19, 20].

We begin with part (i) of Lemma 2.1. The essential step is to establish the estimate

$$|\Pi_h u|^2_{H^1(k)} \le C \left(\sum |\nabla u|_\tau|_2^2 \right) |k|, \quad \forall u \in V_{H_c}, \tag{13}$$

where $k \in \Omega_h$, and the summation is taken over all $\tau \in \Omega_{H_c}$ that have non-empty intersection with k. We note that $\nabla u|_\tau$ is a constant vector(since u is linear in τ) and $|\cdot|_2$ is the usual Euclidean norm in R^d.

If k belongs completely to a single τ, then (13) is obviously true. Otherwise, we denote by $A_i,\ i = 1, \cdots, d+1$, the vertices of k, and it is known that

$$|\Pi_h u|^2_{H^1(k)} \le C \sum_{i,j=1, i<j}^{d+1} \left(u(A_i) - u(A_j) \right)^2 h_k^{d-2}. \tag{14}$$

Here h_k is the diameter of k. Let A_iA_j be the line segment connecting points A_i and A_j. We assume that A_iA_j is cut into l pieces by the coarse tetrahedra $\tau_1^{ij}, \cdots, \tau_l^{ij}$, and $u(\cdot)$ is linear on each piece. By the assumption made at the beginning of Section 2, l is finite. Therefore, by using the triangle inequality and the mean value theorem, we have

$$\left(u(A_i) - u(A_j) \right)^2 \le 2 \sum_{m=1}^{l} \left| \nabla u|_{\tau_m^{ij}} \right|_2^2 h_k^2. \tag{15}$$

(13) can thus be proved by combining the estimates (15) and (14). For $\tau \in \Omega_{H_c}$, we denote by $\tau_j, j = 1, \cdots, l_1$, all the coarse tetrahedra that share at least one of the fine tetrahedra with τ (i.e., this fine tetrahedron intersects with both τ and τ_j). l_1 is a finite number. By summing (13) over all $k_i \in \Omega_h, i = 1, \cdots, m$, whose intersection with τ is non-empty, we obtain

$$|\Pi_h u|^2_{H^1(\tau)} \le \sum_{i=1}^{m} |\Pi_h u|^2_{H^1(k_i)} \le C \sum_{j=1}^{l_1} \left| \nabla u|_{\tau_j} \right|_2^2 \ |\tau_j|. \tag{16}$$

Here we used that fact that, for each τ_j, the sum of the areas of the fine tetrahedra that intersect with τ_j is less than $C\,|\tau_j|$, because of the assumption $\bar{h} \le C\,H_c$. The proof for part (i) of Lemma 2.1 follows immediately by summing (16) over all τ in Ω_{H_c} (the number of repetitions, for each τ, in the summation is finite).

We now turn to the proof of part (ii) of Lemma 2.1. Let k be a fine tetrahedron and A one of its vertices, which implies that $w(A) = 0$. Here $w = u - \Pi_h u$. We estimate the integral

$$\|w\|^2_{L^2(k)} = \int_k w^2(X) d\Omega = \int_k (w(X) - w(A))^2 d\Omega. \tag{17}$$

Let XA be the line segment connecting points X and A. Using the same argument as before, we assume XA is cut into l_2 pieces by coarse tetrahedra $\tau_1^k, \cdots, \tau_{l_2}^k$. Since w is linear on each piece, we have

$$(18) \qquad (w(X) - w(A))^2 \leq 2 \sum_{m=1}^{l_2} \left| \nabla w|_{\tau_m^k} \right|_2^2 h_k^2.$$

By combining the results of (17), (18) and assumptions made at the beginning Section 2, we arrive at

$$(19) \qquad \|w\|_{L^2(k)}^2 \leq C h_k^2 \sum |\nabla w|_{L^2(\tau)}^2,$$

where the sum is taken over all coarse tetrahedra which intersect with k. Summing (19) over all k and noting that the number of coarse tetrahedra overlap with each k is finite, we obtain

$$\|w\|_{L^2(\Omega)}^2 \leq C \bar{h}^2 |\nabla w|_{L^2(\Omega)}^2.$$

Thus, the proof of part (ii) of Lemma 2.1 follows immediately by using the result of part (i).

5. An additive Schwarz method based on a non-nested coarse space. In this section, we define and analyze an additive Schwarz algorithm for solving the finite element problem (1). For $1 \leq i \leq N$, we define the operator $P_i : V_h \rightarrow V_i$ by $a(P_i u, \phi) = a(u, \phi)$, for any $u \in V_h$ and $\phi \in V_i$, and define $P_0' : V_h \rightarrow V_{H_c}$ by $a(P_0' u, \phi) = a(u, \Pi_h \phi)$, for any $u \in V_h$ and $\phi \in V_{H_c}$. Let $P_0 = \Pi_h P_0'$. Similar operators were used in the context of non-nested subspaces based multigrid methods, see e.g., [1, 17]. Let P be defined by $P = \sum_{i=0}^N P_i$. It can be seen easily that P is symmetric in the inner product $a(\cdot, \cdot)$. Let $g = \Pi_h g_0 + \sum_{i=1}^N g_i$ and $g_i = P_i u_h^*$. Following the Schwarz theory of Dryja and Widlund [13], it can be shown that, if the operator P is nonsingular, then the linear operator equation

$$(20) \qquad P u_h^* = g$$

has the same solution as that of (1). We show in the next theorem that P is not only nonsingular but also uniformly bounded from both above and below. A proof can be obtained by using the standard techniques in [13].

Theorem 5.1. *The following estimate holds,*

$$c a(u, u) \leq a(Pu, u) \leq C a(u, u), \quad \forall u \in V_h,$$

where $c = \max\{1/[C_0 (1 + H_c^2/\delta^2)], 1/[C_0 (1 + H/\delta + H_c^2/\delta H)]\}$.

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