Finite Difference Methods for Hyperbolic PDEs

MATH 3014
Monday & Thursday 14:30-15:45
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For the second-order canonical form

$$a(x,t)u_{tt} + 2b(x,t)u_{xt} + c(x,t)u_{xx} + \text{lower-order terms} = f(x,t)$$

is hyperbolic if $b^2 - ac > 0$ in the entire x and t domain.

A few typical model problems involving hyperbolic PDEs are as follows:

- Second-order linear wave equation:

$$u_{tt} = a u_{xx}\,, \qquad 0 < x < 1\,,$$

$$u(x,0) = \eta(x), \qquad \frac{\partial u}{\partial t}(x,0) = v(x), \qquad \text{IC}\,,$$

$$u(0,t) = g_l(t)\,, \qquad u(1,t) = g_r(t), \qquad \text{BC}.$$

- Advection equation (one-way wave equation):

$$u_t + au_x = 0\,, \quad 0 < x < 1\,,$$
$$u(x,0) = \eta(x)\,, \qquad \text{IC}\,,$$
$$u(0,t) = g_l(t) \quad \text{if} \quad a \geq 0\,, \quad \text{or} \quad u(1,t) = g_r(t) \quad \text{if} \quad a \leq 0\,.$$

Here g_l and g_r are prescribed boundary conditions from the left and right, respectively.

- Linear first-order hyperbolic system:

$$\mathbf{u}_t = A\mathbf{u}_x + \mathbf{f}(x,t)\,,$$

where **u** and **f** are two vectors and A is a matrix. The system is called hyperbolic if A is diagonalizable, i.e., if there is a nonsingular matrix T such that $A = TDT^{-1}$, and all eigenvalues of A are real numbers.

- Nonlinear hyperbolic equation or system, notably conservation laws:

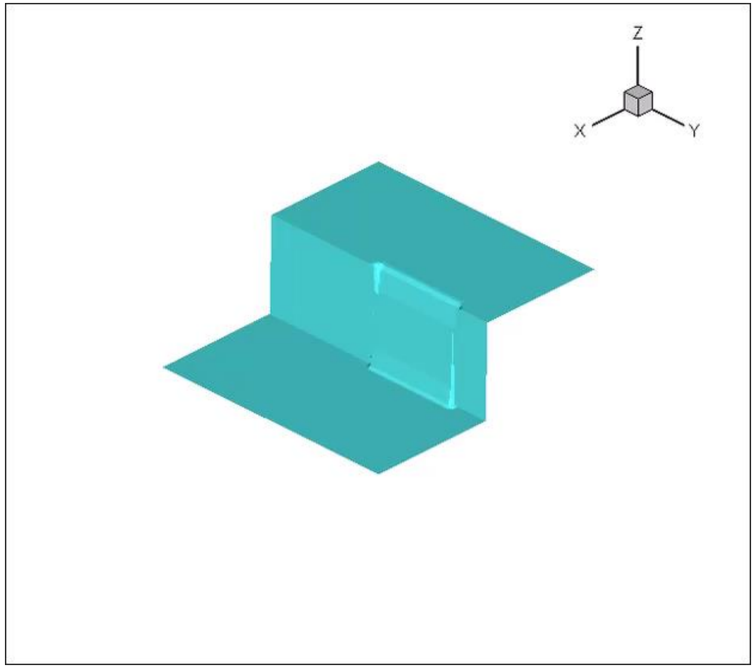
$$u_t + f(u)_x = 0\,, \quad e.g.,\ \text{Burgers' equation}\ u_x + \left(\frac{u^2}{2}\right)_x = 0\,;$$

$$\mathbf{u}_t + \mathbf{f}_x + \mathbf{g}_y = 0\,.$$

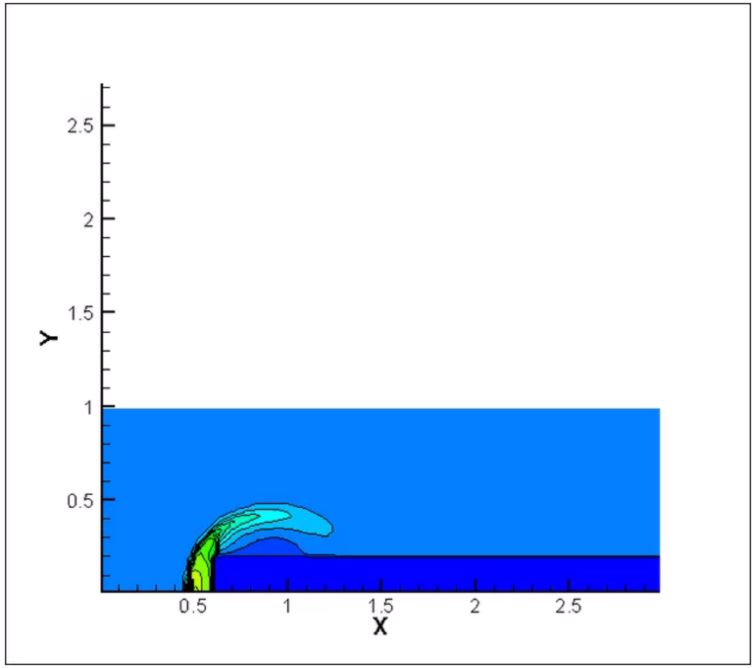
For nonlinear hyperbolic PDE, shocks (a discontinuous solution) can develop even if the initial data is smooth.

Some phenomena of shocks

Sound barrier



Dam breaking



Forward-facing step flow

5.1 Characteristics and Boundary Conditions

The exact solution for the one-way wave equation

$$u_t + au_x = 0\,, \quad -\infty < x < \infty\,,$$

$$u(x,0) = \eta(x)\,, \quad t > 0$$

is $u(x,t) = \eta(x - at)$.

For the finite domain problem

$$u_t + au_x = 0\,, \quad 0 < x < 1\,,$$

$$u(x,0) = \eta(x)\,, \quad t > 0\,, \qquad \boxed{u(0,t) = g_l(t) \quad \text{if } a > 0}$$

We consider the method of characteristics in which the solution is constant along the characteristics.

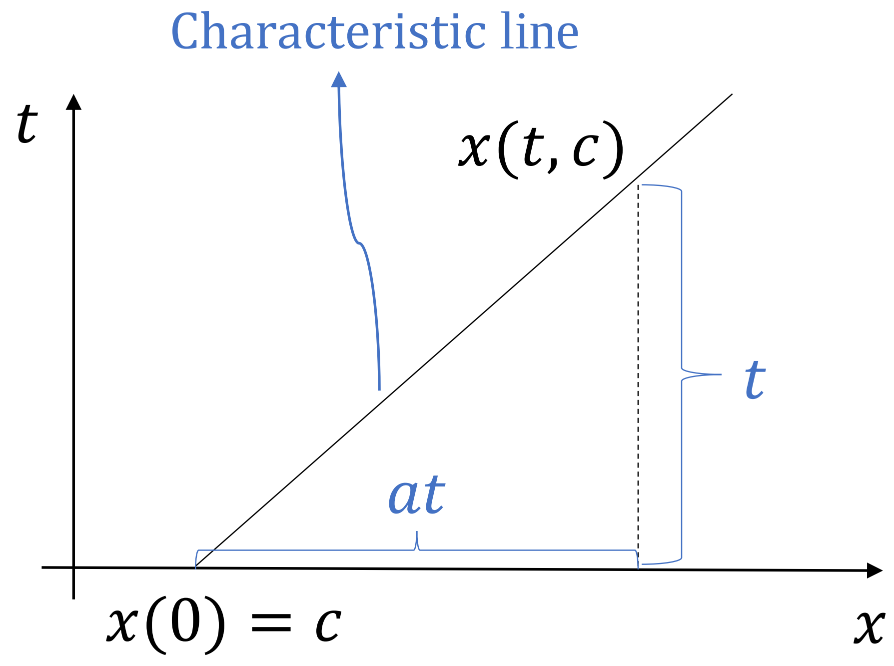
The method of characteristics

For the IVP of ODE
$$\begin{cases} \dfrac{dx}{dt} = a \\ x(0) = c \end{cases}$$

The solution $x(t,c) = at + c$ is called the characteristic line of the IVP of PDE.

Along the characteristic line $x = x(t,c)$, $u = (x(t,c), t)$ satisfies the following ODE

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0,$$

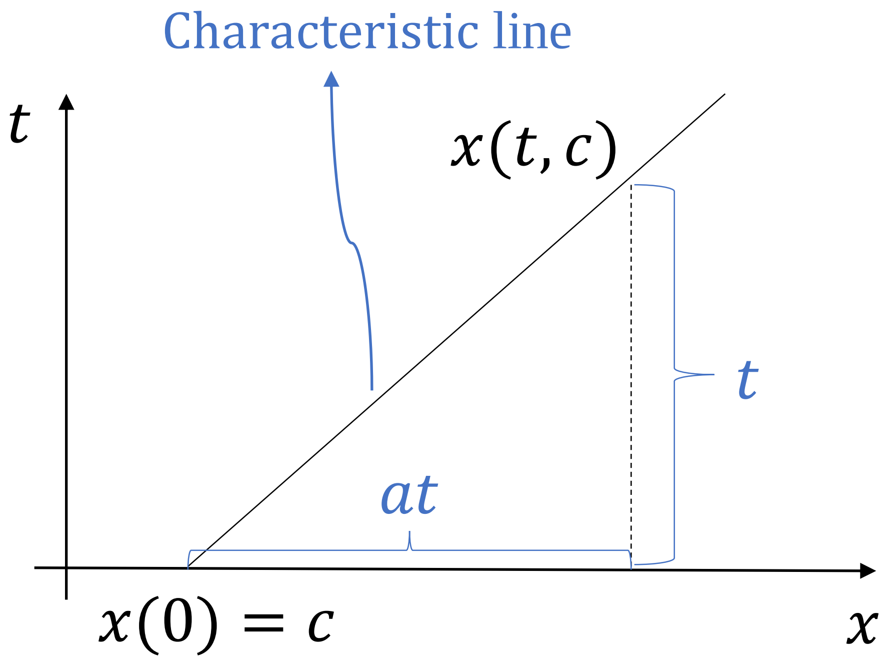


The $x - t$ diagram

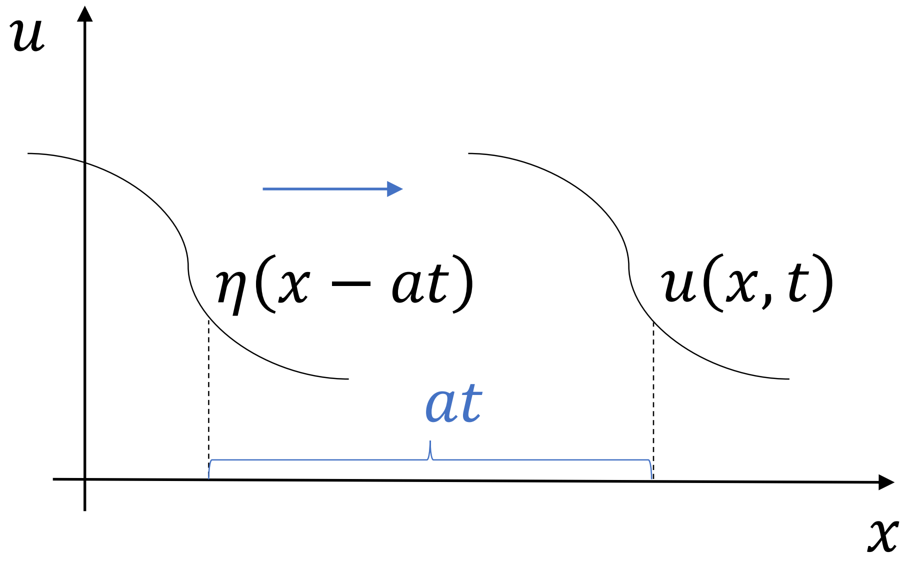
which means that $u(x,t)$ is constant along the characteristic line. Therefore, we have

$$u(x(t,c),t) = u(x(0,c),0) = u(c,0) = \eta(c)$$

$$\Rightarrow \quad u(x,t) = \eta(x - at).$$



The $x - t$ diagram



The $x - u$ diagram

5.2 Finite Difference Schemes for Hyperbolic equations

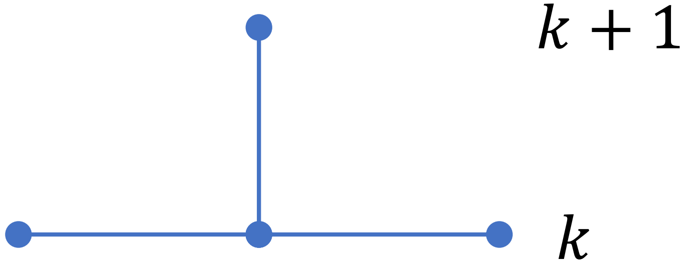
- Lax–Friedrichs method;
- Upwind scheme;
- Leap-frog method;
- Lax–Wendroff method;
- Crank–Nicolson scheme; and
- Beam–Warming method.

Consider the one-way wave equation $u_t + a u_x = 0$

The FW-CT scheme:

$$U_j^{k+1} = U_j^k - \mu \left(U_{j+1}^k - U_{j-1}^k \right), \qquad \mu = a\Delta t/(2h).$$

- The local truncation error is $O(\Delta t + h^2)$
- The method is unconditionally unstable



The von Neumann stability analysis:

(The growth factor) $$g(\theta) = 1 - \mu \left(e^{ih\xi} - e^{-ih\xi} \right)$$

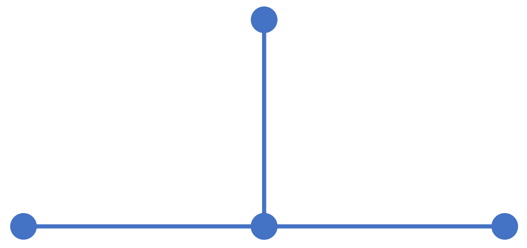
$$= 1 - \mu\, 2\, i \sin(h\xi), \text{ where } \theta = h\xi,\ \mu = a\Delta t/(2h).$$

we have

$$|g(\theta)|^2 = 1 + 4\mu^2 \sin^2(h\xi) \geq 1 .$$

The Lax–Friedrichs scheme: average U_j^k using U_{j-1}^k and U_{j+1}^k to get

$$U_j^{k+1} = \frac{1}{2}\left(U_{j-1}^k + U_{j+1}^k\right) - \mu\left(U_{j+1}^k - U_{j-1}^k\right).$$



- The local truncation error is $O(\Delta t + h)$ if $\Delta t \simeq h$.
- The method is conditionally stable

The growth factor:

$$g(\theta) = \frac{1}{2}\left(e^{ih\xi} + e^{-ih\xi}\right) + \mu\left(e^{ih\xi} - e^{-ih\xi}\right)$$
$$= \cos(h\xi) - 2\mu\sin(h\xi)i$$

$$\begin{aligned} |g(\theta)|^2 &= \cos^2(h\xi) + 4\mu^2\sin^2(h\xi) \\ &= 1 - \sin^2(h\xi) + 4\mu^2\sin^2(h\xi) \\ &= 1 - (1 - 4\mu^2)\sin^2(h\xi), \end{aligned}$$

Therefore, if $\Delta t \le h/|a|$ ⇒ $1 - 4\mu^2 \ge 0$ ⇒ $|g(\theta)| \le 1$

The Lax–Friedrichs scheme

```
clear; close all

a = 0; b=1; tfinal = 0.5; m=20;
h = (b-a)/m;
k = h; mu = k/h;

t = 0;
n = fix(tfinal/k);
y1 = zeros(m+1,1); y2=y1; x=y1;
% Initial condition
for i=1:m+1,
    x(i) = a + (i-1)*h;
    y1(i) = uexact(t,x(i)); % current
    y2(i) = 0; % next level
end
```

```
t = 0;
for j=1:n,
    y1(1)=bc(t);
    y2(1)=bc(t+k); % Physical boundary condition
    for i=2:m
        % FW-CT scheme
        % y2(i) = y1(i) - mu*(y1(i+1)-y1(i-1))/2;
        % Lax-Friderichs scheme
        y2(i) = 0.5*(y1(i+1)+y1(i-1)) - mu*(y1(i+1)-y1(i-1))/2;
    End

    i = m+1;
    % Numerical boundary condition
    y2(i) = y1(i) - mu*(y1(i)-y1(i-1) );

    t = t + k;
    y1 = y2;
    plot(x,y2); pause(0.5)

end
```

Problem: the 1D advection equation

$$u_t + u_x = 0 \text{ in the domain } 0 < x < 1.$$

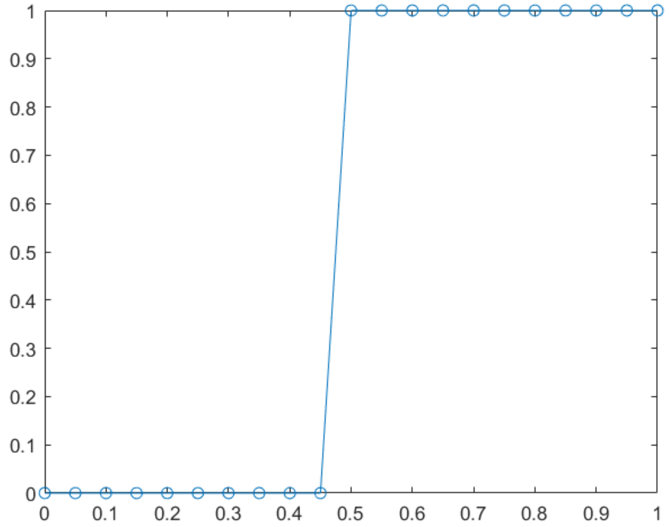
The initial condition is $$u(x,0) = u_0(x) = \begin{cases} 0 & \text{if } 0 < x < 1/2, \\ 1 & \text{if } 1/2 \le x < 1. \end{cases}$$

The boundary condition is $u(0,t) = \sin t$.

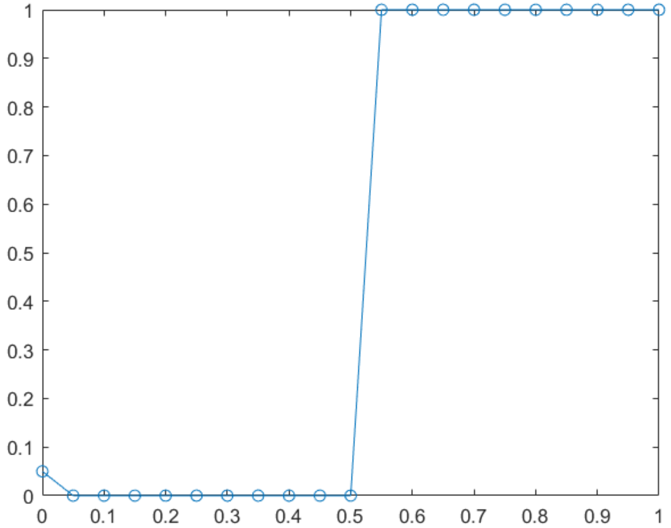
The analytic solution is $$u(x,t) = \begin{cases} u_0(x-t) & \text{if } 0 < t < x < 1, \\ \sin(t-x) & \text{if } 0 < x < t < 1. \end{cases}$$

$t - x$ is the time period that the sin function experiences.

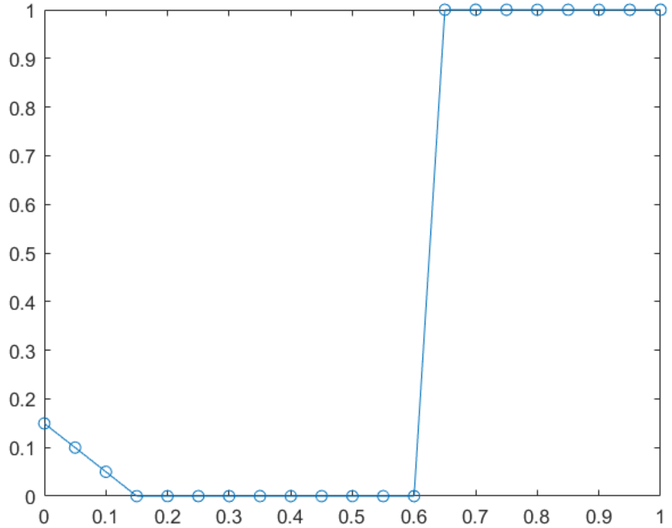
Solutions at different time steps obtained using the Lax–Friedrichs scheme when $\Delta t = h$



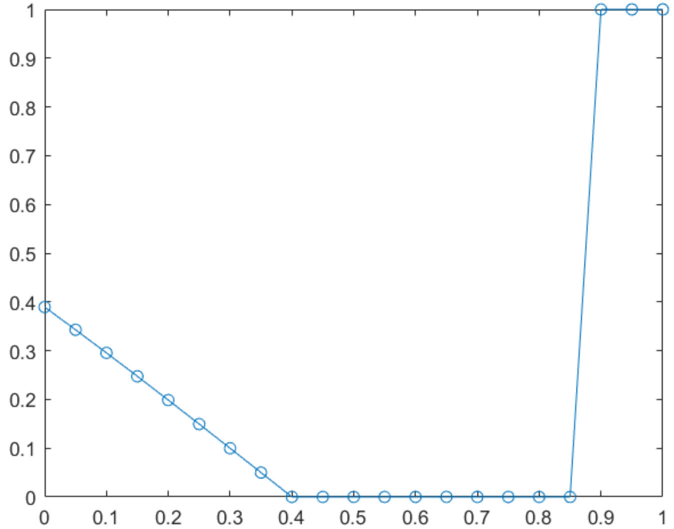
Time step 0 (initial condition)



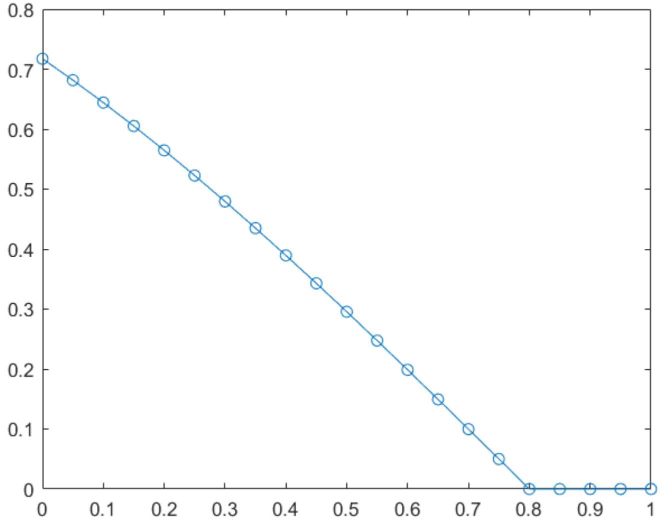
Time step 1



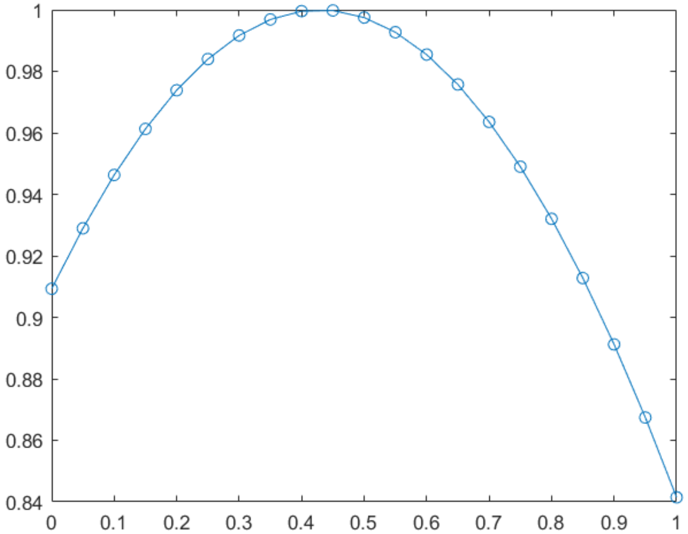
Time step 3



Time step 8



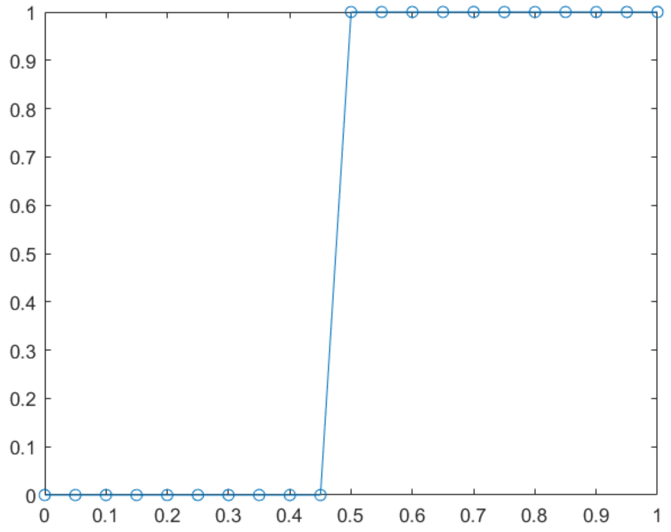
Time step 16



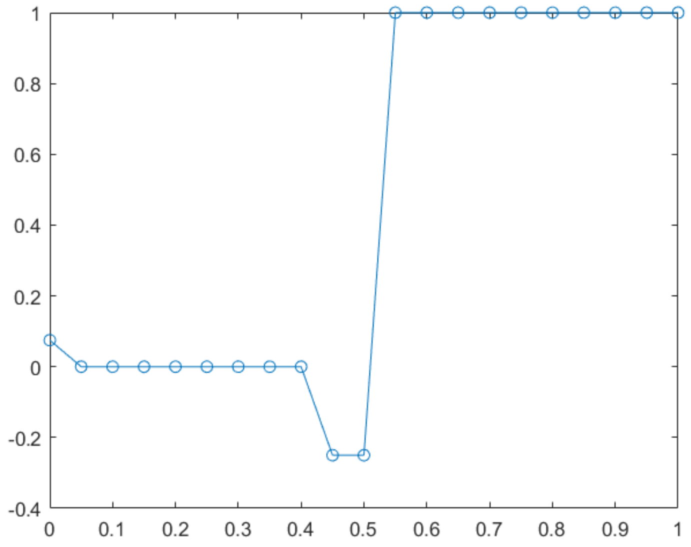
Time step 40

Solutions at different time steps obtained using the Lax–Friedrichs scheme

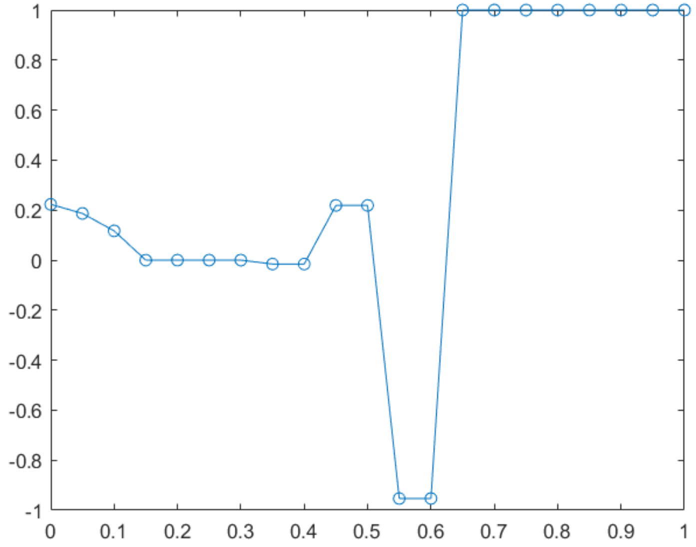
when $\Delta t = 1.5h$ (Blow up)



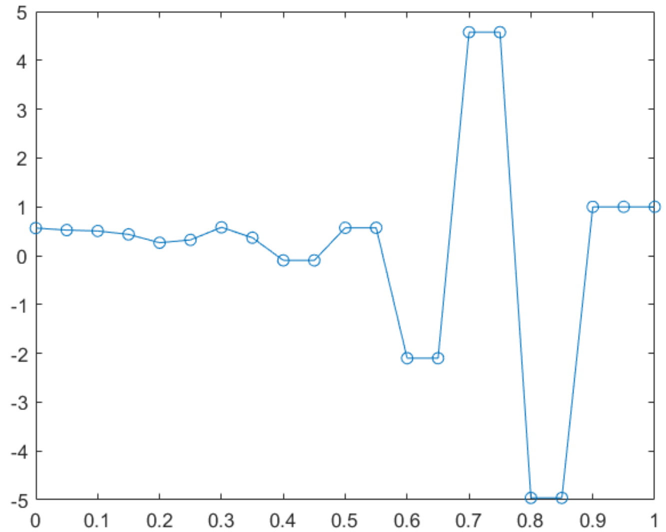
Time step 0 (initial condition)



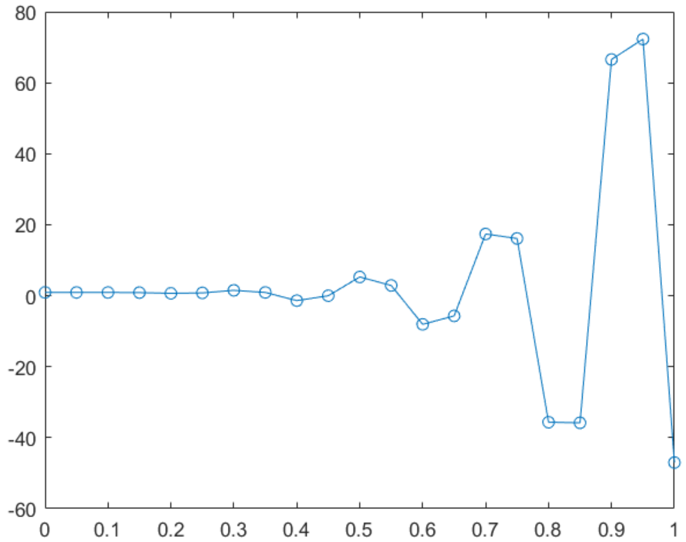
Time step 1



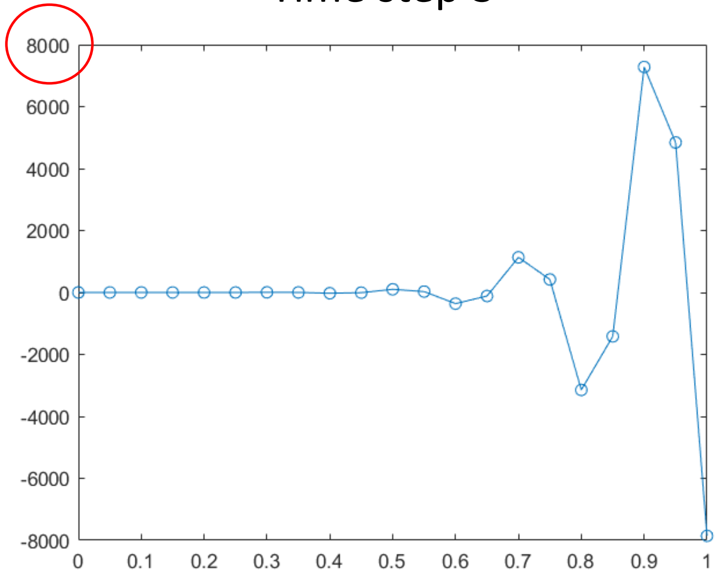
Time step 3



Time step 8



Time step 16

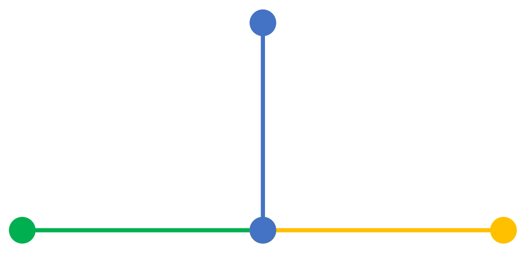


Time step 40

The Upwind scheme:

The upwind scheme for $u_t + au_x = 0$ is

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} = \begin{cases} -\frac{a}{h}\left(U_j^k - U_{j-1}^k\right) & \text{if } a \geq 0\,, \\ -\frac{a}{h}\left(U_{j+1}^k - U_j^k\right) & \text{if } a < 0\,, \end{cases}$$



- The scheme is first-order accurate in time and in space
- The method is conditionally stable

The growth factor for the case when $a \geq 0$ is:

$$\begin{aligned} g(\theta) &= 1 - \mu\left(1 - e^{-ih\xi}\right) \\ &= 1 - \mu(1 - \cos(h\xi)) - i\mu\sin(h\xi) \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} |g(\theta)|^2 &= (1 - \mu + \mu\cos(h\xi))^2 + \mu^2\sin^2(h\xi) \\ &= (1-\mu)^2 + 2(1-\mu)\mu\cos(h\xi) + \mu^2 \\ &= 1 - 2(1-\mu)\mu(1 - \cos(h\xi))\,, \end{aligned}$$

so if $1 - \mu \geq 0$ (*i.e.*, $\mu \leq 1$) or $\Delta t \leq h/a$ we have $|g(\theta)| \leq 1$.

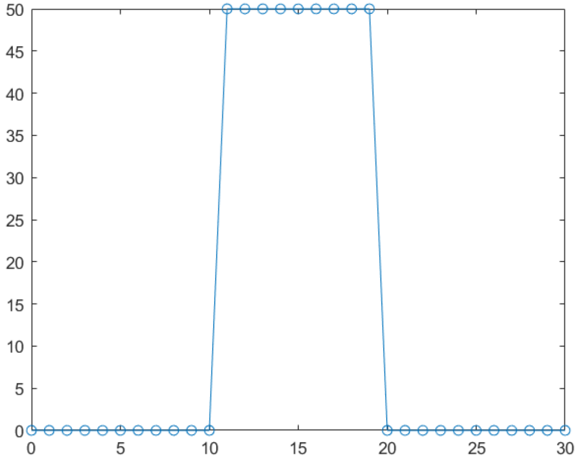
The Upwind scheme

```
clear; close all

a = 0; b=1; tfinal = 0.5; m = 20;

aa = 1; % The coefficient
h = (b-a)/m; k = h/abs(aa);
mu = aa*k/h; % Set mesh and time step.

t = 0; n = fix(tfinal/k);
y1 = zeros(m+1,1); y2=y1; x=y1;

figure(1);
%axis([-0.1 1.1 -0.1 1.1]);
for i=1:m+1,
      x(i) = a + (i-1)*h;
      y1(i) = uexact(t,x(i)); % Initial data
      y2(i) = 0;
end
```

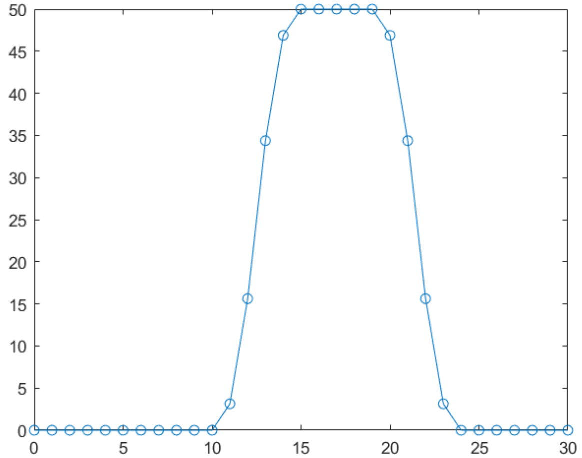
```
% Time marching
 for j=1:n,
    y1(1)=bc(t);
    y2(1)=bc(t+k);
    for i=2:m+1
       y2(i) = y1(i) - mu*(y1(i)-y1(i-1) );
    end
    t = t + k;
    y1 = y2;
    plot(x,y2); pause(0.5);
End

% Define exact solution for comparison
 u_e = zeros(m+1,1);
 for i=1:m+1
    u_e(i) = uexact(t,x(i));
 end

 max(abs(u_e-y2))
 plot(x,y2,'o',x,u_e)
```

Solutions at different time steps obtained using the Upwind scheme

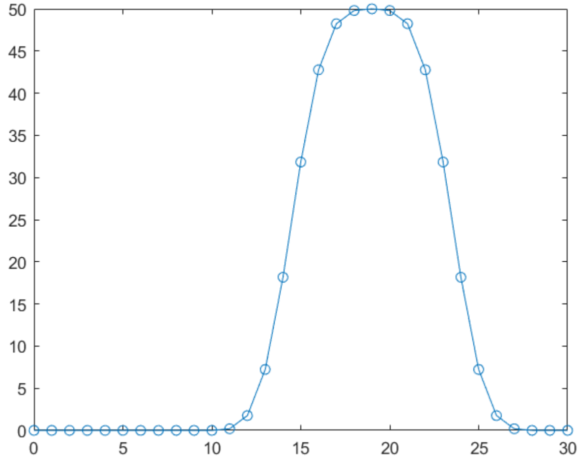
when $\Delta t = 0.5h$ (Smooth out effect)



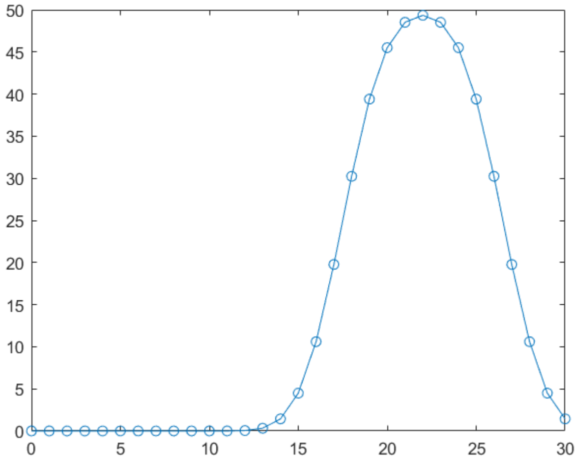
Time step 0 (initial condition)



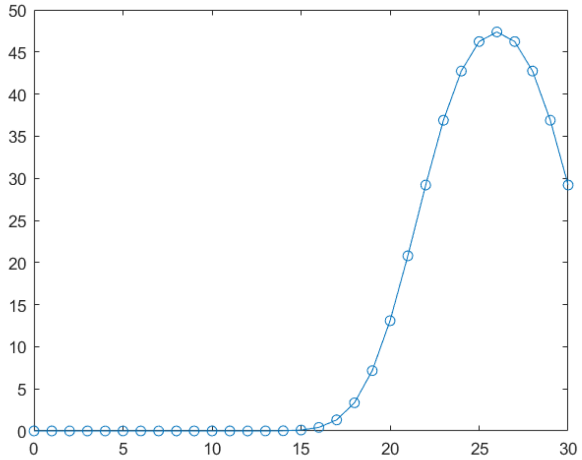
Time step 2



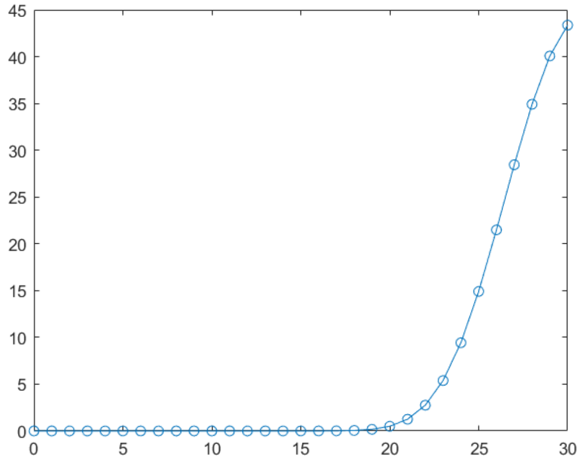
Time step 4



Time step 6



Time step 8



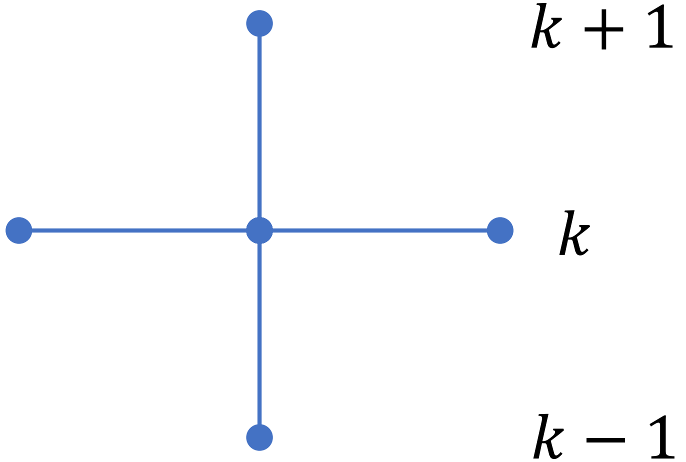
Time step 10

The Leap-frog Scheme:

The leap-frog scheme for $u_t + au_x = 0$ is

$$\frac{U_j^{k+1} - U_j^{k-1}}{2\Delta t} + \frac{a}{2h}\left(U_{j+1}^k - U_{j-1}^k\right) = 0\,,$$

or $$U_j^{k+1} = U_j^{k-1} - \mu\left(U_{j+1}^k - U_{j-1}^k\right),$$



- The discretization is second-order in time and in space.
- The method is conditionally stable, CFL condition: $\Delta t < \frac{h}{|a|}$.
- It requires an Numerical Boundary Condition at one end.
- It needs U_j^1 to get started, we can use the upwind or other scheme to obtain U_j^1.

The von Neumann analysis for the leap scheme

Substituting

$$U_j^k = e^{ij\xi}, \quad U_j^{k+1} = g(\xi) e^{ij\xi}, \quad U_j^{k-1} = \boxed{\frac{1}{g(\xi)}} e^{ij\xi}$$

into the leap-frog scheme, we get

$$g^2 + \mu(e^{ih\xi} - e^{-ih\xi})g - 1 = 0\,,$$

$$\text{or} \qquad g^2 + 2\mu i \sin(h\xi)\, g - 1 = 0\,,$$

with solution

$$g_{\pm} = -i\mu \sin(h\xi) \pm \sqrt{1 - \mu^2 \sin^2(h\xi)}\,. \tag{5.10}$$

We distinguish three different cases.

1. If $|\mu| > 1$, then there are ξ such that at least one of $|g_-| > 1$ or $|g_+| > 1$ holds, so the scheme is unstable!
2. If $|\mu| < 1$, then $1 - \mu^2 \sin^2(h\xi) \geq 0$ such that

$$|g_{\pm}|^2 = \mu^2 \sin^2(h\xi) + 1 - \mu^2 \sin^2(h\xi) = 1\,.$$

3. If $|\mu| = 1$, we still have $|g_{\pm}| = 1$, but we can find ξ such that $\mu \sin(h\xi) = 1$ and $g_+ = g_- = -i$, *i.e.*, $-i$ is a double root of the characteristic polynomial. The solution of the finite difference equation therefore has the form

$$U_j^k = C_1(-i)^k + C_2 k(-i)^k ,$$

where the possibly complex numbers C_1 and C_2 are determined from the initial conditions. Thus there are solutions such that $\|\mathbf{U}^k\| \simeq k$ which are unstable (slow growing).

5.3 The Modified PDE and Numerical Diffusion/Dispersion

A modified PDE is the PDE that a finite difference equation satisfies exactly at grid points.

Take the upwind method for the advection equation $u_t + au_x = 0$ with $a > 0$ for example

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + \frac{a}{h}\left(U_j^k - U_{j-1}^k\right) = 0\,.$$

The derivation of a modified PDE is similar to computing the local truncation error.

Insert $v(x,t)$ in to the finite difference equation to derive a PDE that $v(x,t)$ satisfies exactly.

$$\frac{v(x.t + \Delta t) - v(x,t)}{\Delta t} + \frac{a}{h}\left(v(x,t) - v(x-h,t)\right) = 0\,.$$

$$\frac{v(x.t+\Delta t)-v(x,t)}{\Delta t}+\frac{a}{h}(v(x,t)-v(x-h,t))=0\,.$$

Expanding the terms in Taylor series about (x,t) and simplifying yields

$$v_t+\frac{1}{2}\Delta t\,v_{tt}+\cdots+a\left(v_x-\frac{1}{2}hv_{xx}+\frac{1}{6}h^2v_{xxx}+\cdots\right)=0\,,$$

which can be rewritten as

The modified PDE

$$v_t+av_x=\frac{1}{2}(ahv_{xx}-\Delta t v_{tt})-\frac{1}{6}\left(ah^2v_{xxx}+(\Delta t)^2v_{tt}\right)+\cdots\,,$$

High order terms can be ignored

which is the PDE that v satisfies. Consequently,

$\frac{\partial}{\partial t}$ on both sides

$$\begin{aligned}v_{tt}&=-av_{xt}+\frac{1}{2}(ahv_{xxt}-\Delta t v_{ttt})\\&=-av_{xt}+O(\Delta t,h)\\&=-a\frac{\partial}{\partial x}\left(-av_x+O(\Delta t,h)\right),\end{aligned}$$

Use $v_t+av_x=\frac{1}{2}(ahv_{xx}-\Delta t v_{tt})+O(\Delta t,h)$

so the leading modified PDE is

$$v_t+av_x=\frac{1}{2}ah\left(1-\frac{a\Delta t}{h}\right)v_{xx}\,. \qquad (5.11)$$

Advection–diffusion equation

The original PDE

$$u_t + au_x = 0\,,\ a > 0$$

A first-order accurate approximation to the true solution of the original PDE

The upwind scheme

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + \frac{a}{h}\left(U_j^k - U_{j-1}^k\right) = 0$$

The high order terms are $O(h^2 + \Delta t^2) + O(\Delta t^2 h)$

Satisfies exactly

A second-order accurate approximation to the true solution of the leading modified PDE

The modified PDE

$$v_t + av_x = \frac{1}{2}(ahv_{xx} - \Delta t v_{tt}) - \frac{1}{6}\left(ah^2 v_{xxx} + (\Delta t)^2 v_{tt}\right) + \cdots$$

The leading modified PDE

$$v_t + av_x = \frac{1}{2}ah\left(1 - \frac{a\Delta t}{h}\right)v_{xx}$$

The modified equation tells some features of the scheme:

- The computed solution smooths out discontinuities because of the diffusion term $\frac{1}{2}ah\left(1-\frac{a\Delta t}{h}\right)v_{xx}$
- We have second-order accuracy to $u_t + au_x = 0$ if a is a constant and $\Delta t = h/a$.
- We can add the correction term to offset the leading error term to render a higher-order accurate method, but the stability needs to be checked. For instance, we can modify the upwind scheme to get a second-order scheme when $\Delta t \simeq h$:

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + a\frac{U_j^k - U_{j-1}^k}{h} = \frac{1}{2}ah\left(1-\frac{a\Delta t}{h}\right)\frac{U_{j-1}^k - 2U_j^k + U_{j+1}^k}{h^2}$$

this approximates u_{xx} with $O(h^2)$, therefore the RHS of (5.11) can be cancelled with $O(h^3)$.

Why some schemes are unstable?

--- check the modified equation

The PDE: $u_t + au_x = 0$

The FW-CT scheme: $$\frac{U_j^{k+1} - U_j^k}{\Delta t} + a\frac{U_{j+1}^k - U_{j-1}^k}{2h} = 0$$

$$\frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} = u_t + \frac{\Delta t}{2}u_{tt} + O((\Delta t)^2)$$

$$= u_t + \frac{1}{2}a^2(\Delta t)u_{xx} + O((\Delta t)^2)\ .$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(-a\frac{\partial u}{\partial x}\right) = -a\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = -a\frac{\partial}{\partial x}\left(-a\frac{\partial u}{\partial x}\right)$$

The leading term of the modified PDE for the FW-CT scheme:

$$v_t + av_x = -\frac{a^2\Delta t}{2}v_{xx}$$

The sign is negative here! Similar to the backward heat equation that is dynamically unstable

5.4 The Lax–Wendroff Scheme

Note that for the time discretization of the PDE $u_t + au_x = 0$:

$$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} = u_t + \frac{\Delta t}{2}u_{tt} + O((\Delta t)^2)$$

$$= u_t + \boxed{\frac{1}{2}a^2(\Delta t)u_{xx}} + O((\Delta t)^2)\,.$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left(-a\frac{\partial u}{\partial x}\right) = -a\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = -a\frac{\partial}{\partial x}\left(-a\frac{\partial u}{\partial x}\right)$$

Recall that $T(x) = \dfrac{u(x-h)-2u(x)+u(x+h)}{h^2} - u_{xx}(x) = \dfrac{h^2}{12}u^{(4)}(x) + \cdots = O(h^2)$

Hence $\frac{1}{2}a^2(\Delta t)u_{xx}$ can be cancelled by using the central finite difference with 2nd order accuracy

The Lax–Wendroff scheme:

One additional term compared to FW-CT

$$\frac{U_j^{k+1}-U_j^k}{\Delta t} + a\frac{U_{j+1}^k - U_{j-1}^k}{2h} = \boxed{\frac{1}{2}\frac{a^2\Delta t}{h^2}\left(U_{j-1}^k - 2U_j^k + U_{j+1}^k\right)}, \qquad (5.14)$$

The derivation of the L-W scheme is easier than the derivation of the modified upwind second-order scheme on Page 21, because the central difference for the first-order term au_x already gives a high order truncation error, here we only do the Taylor expansion w.r.t t.

To derive the Lax-Wendroff scheme

1. Do the Taylor expansion only with respect to t.

2. Make use of the original PDE to transform u_{tt} to a term involving the derivatives w.r.t x (i.e., u_{xx}), the resulting formulation is called the modified PDE.

3. Apply finite difference for the term involving the derivatives w.r.t x (Spatial discretization).

The Lax–Wendroff scheme is second-order accurate both in time and space.

The local truncation error of the Lax–Wendroff scheme:

$$T(x,t) = \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} - \frac{a\,(u(x+h,t) - u(x-h,t))}{2h}$$
$$- \frac{a^2 \Delta t\,(u(x-h,t) - 2u(x,t) + u(x+h,t))}{2h^2}$$
$$= u_t + \frac{\Delta t}{2} u_{tt} - a u_x - \frac{a^2 \Delta t}{2} u_{xx} + O((\Delta t)^2 + h^2)$$
$$= O((\Delta t)^2 + h^2)\,,$$

$$u_t = -a u_x$$

$$u_{tt} = -a u_{xt} = -a \frac{\partial}{\partial x} u_t = a^2 u_{xx}$$

The CFL condition for the Lax–Wendroff scheme

The von Neumann stability analysis

$$g(\theta) = 1 - \frac{\mu}{2}\left(e^{ih\xi} - e^{-ih\xi}\right) + \frac{\mu^2}{2}\left(e^{-ih\xi} - 2 + e^{ih\xi}\right)$$
$$= 1 - \mu i \sin\theta - 2\mu^2 \sin^2(\theta/2),$$

where again $\theta = h\xi$, so

$$|g(\theta)|^2 = \left(1 - 2\mu^2 \sin^2\frac{\theta}{2}\right)^2 + \mu^2 \sin^2\theta$$
$$= 1 - 4\mu^2 \sin^2\frac{\theta}{2} + 4\mu^4 \sin^4\frac{\theta}{2} + 4\mu^2 \sin^2\frac{\theta}{2}\left(1 - \sin^2\frac{\theta}{2}\right)$$
$$= 1 - 4\mu^2\left(1 - \mu^2\right)\sin^4\frac{\theta}{2}$$
$$\leq 1 - 4\mu^2\left(1 - \mu^2\right).$$

If this is positive, then stable

We conclude $|g(\theta)| \leq 1$ if $\mu \leq 1$, *i.e.*, $\Delta t \leq h/|a|$. If $\Delta t > h/|a|$, there are ξ such that $|g(\theta)| > 1$ so the scheme is unstable.

Advantages of the Lax–Wendroff scheme:

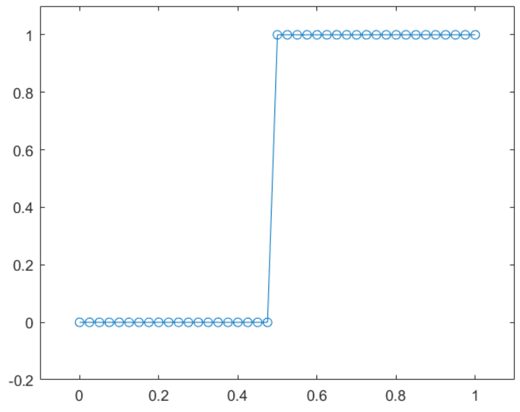
- Second-order accurate both in time and space
- Conditionally stable ($\Delta t \leq h/|a|$)

Disadvantages of the Lax–Wendroff scheme:

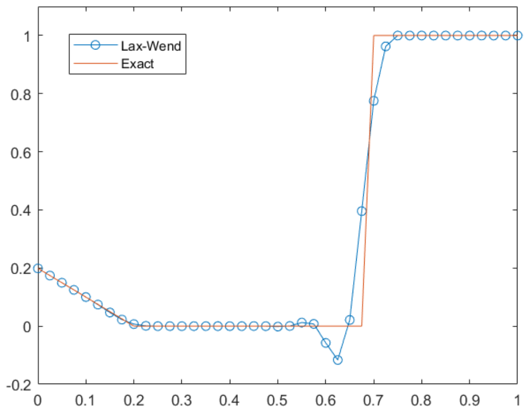
- Leads to a dispersive modified PDE

$$v_t + a v_x = -\frac{1}{6} a h^2 \left(1 - \left(\frac{a\Delta t}{h}\right)^2\right) v_{xxx}$$

- The numerical result can be expected to develop a train of oscillations behind the discontinuity



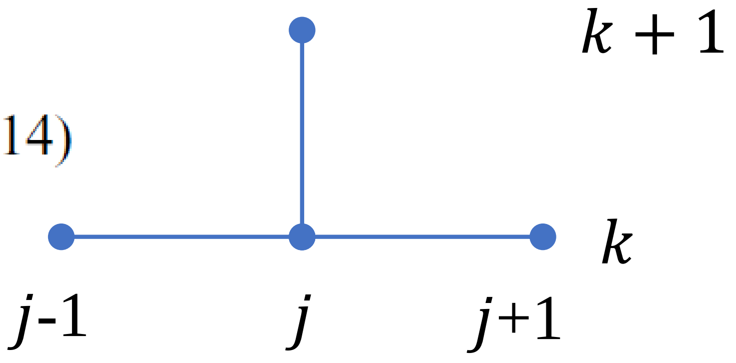
Time step 0 (initial condition)



Time step 10

The Lax–Wendroff method for $u_t + au_x = 0$:

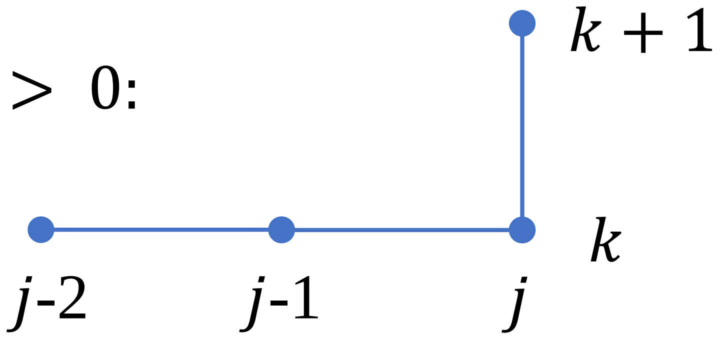
$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + a\frac{U_{j+1}^k - U_{j-1}^k}{2h} = \frac{1}{2}\frac{a^2 \Delta t}{h^2}\left(U_{j-1}^k - 2U_j^k + U_{j+1}^k\right), \quad (5.14)$$



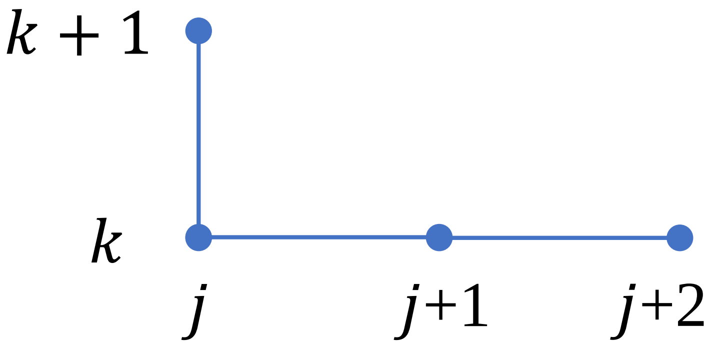
The Beam–Warming method for $u_t + au_x = 0$ for $a > 0$ is

$$U_j^{k+1} = U_j^k - \frac{a\Delta t}{2h}\left(3U_j^k - 4U_{j-1}^k + U_{j-2}^k\right) + \frac{(a\Delta t)^2}{2h^2}\left(U_j^k - 2U_{j-1}^k + U_{j-2}^k\right), \quad (5.18)$$

When $a > 0$:



When $a < 0$:



5.4 The Beam–Warming scheme

- The Beam–Warming method is second-order accurate in time and space if $\Delta t \simeq h$.

  Recall the one-sided finite difference formulas

$$u'(x) = \frac{3u(x) - 4u(x-h) + u(x-2h)}{2h} + O(h^2)\,,$$

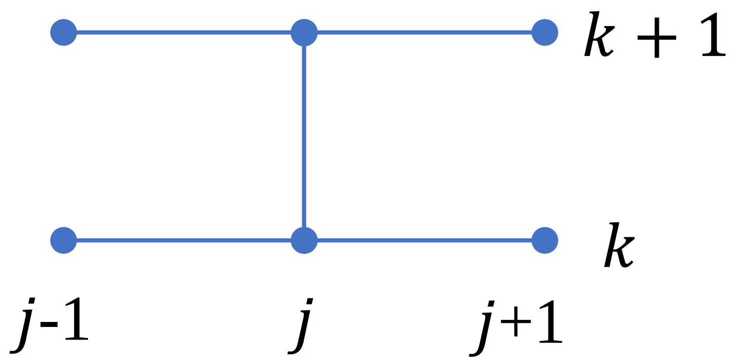
$$u''(x) = \frac{u(x) - 2u(x-h) + u(x-2h)}{h^2} + O(h)\,.$$

- The CFL constraint is $0 < \Delta t \leq \dfrac{2h}{|a|}$.

- For this method, we do not require an Numerical Boundary Condition (NBC) at $x = 1$, but we need a scheme to compute the solution U_1^j.

5.4.2 The Crank–Nicolson Scheme

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + a\frac{U_{j+1}^k - U_{j-1}^k + U_{j+1}^{k+1} - U_{j-1}^{k+1}}{4h} = f_j^{k+\frac{1}{2}}, \qquad (5.21)$$



- Second-order accurate in time and in space.
- Unconditionally stable.
- An NBC is needed at $x = 1$ for case $a > 0$.
- This method is effective for the 1D problem, since it is easy to solve the resulting tridiagonal system of equations.
- For 2D and 3D, use Alternating Direction Implicit (ADI) Method.

5.5 Numerical Boundary Condition

For the one-way wave equation $u_t + au_x = 0$, we need a numerical boundary condition (NBC) at one end when we use any of the "central type" FDM, i.e., the Lax–Friedrichs, Lax–Wendroff, or leapfrog schemes.

- We have one (and only one) physical boundary condition at one end.
- For "upwind type" FDM, we don't need NBC, i.e., the Upwind scheme, the Beam-Warming scheme.

Lagrange interpolation

$$f(x) \simeq f(x_1)\frac{x - x_2}{x_1 - x_2} + f(x_2)\frac{x - x_1}{x_2 - x_1} .$$

First-order approximation: $U_M^{k+1} = U_{M-1}^{k+1}.$

Second-order approximation: $U_M^{k+1} = U_{M-2}^{k+1}\frac{x_M - x_{M-1}}{x_{M-1} - x_M} + U_{M-1}^{k+1}\frac{x_M - x_{M-2}}{x_{M-2} - x_{M-1}} .$

If a uniform grid is used

$$U_M^{k+1} = -U_{M-2}^{k+1} + 2U_{M-1}^{k+1} .$$

5.6 Finite Difference Methods for Second-order Linear Hyperbolic PDEs

Modeling 1D sound wave propagates in two directions

$$u_{tt} = a^2 u_{xx}, \quad \text{where } a > 0 \text{ is the wave speed.}$$

Find the analytic solution by changing variables:

$$\text{Let} \begin{cases} \xi = x - at \\ \eta = x + at \end{cases} \quad \text{or} \quad \begin{cases} x = \dfrac{\xi + \eta}{2} \\ t = \dfrac{\eta - \xi}{2a} \end{cases}$$

Using the chain-rule

$$u_t = -au_\xi + au_\eta \quad u_{tt} = a^2 u_{\xi\xi} - 2a^2 u_{\xi\eta} + a^2 u_{\eta\eta},$$

$$u_x = u_\xi + u_\eta, \quad u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

$$u_{\xi\xi} a^2 - 2a^2 u_{\xi\eta} + a^2 u_{\eta\eta} = a^2 (u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}),$$

$$4a^2 u_{\xi\eta} = 0$$

Find the analytic solution as follows:

$$u_{tt} = a^2 u_{xx},$$

↓ Changing of variables

$$4a^2 u_{\xi\eta} = 0$$

↓ u must not contain any mix terms that depend on both ξ and η, otherwise RHS $\neq 0$

$$u(x,t) = F(\xi) + G(\eta),$$

where $F(\xi)$ and $G(\eta)$ are two differential functions of one variable.

↓

$$u(x,t) = F(x - at) + G(x + at)$$

The two functions F, G are determined by initial and boundary conditions.

Example: the Cauchy problem

- 1D wave propagates in two directions

$$u_{tt} = a^2 u_{xx} , \qquad -\infty < x < \infty ,$$

$$u(x,0) = u_0(x) , \quad u_t(x,0) = g(x) ,$$

The analytic solution is called the D'Alembert's formula, as

$$u(x,t) = \frac{1}{2}\left(u_0(x-at) + u_0(x+at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} g(s)ds.$$

- 1D wave propagates in one direction

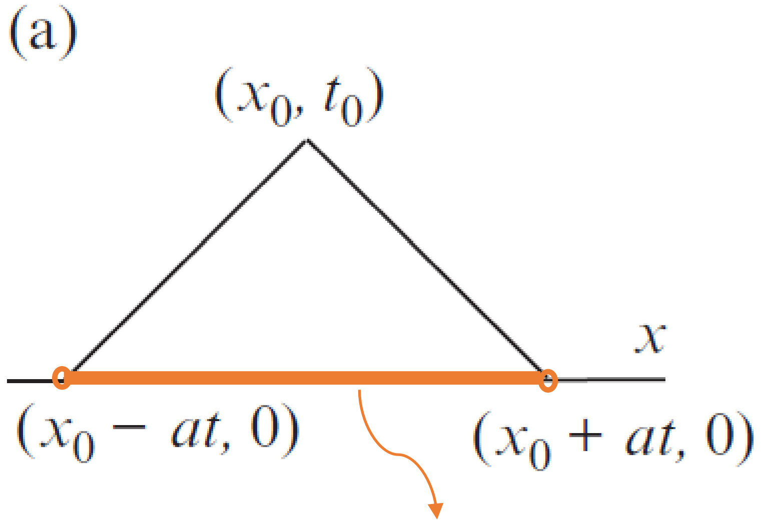
For comparison

$$u_t + au_x = 0 , \quad -\infty < x < \infty ,$$

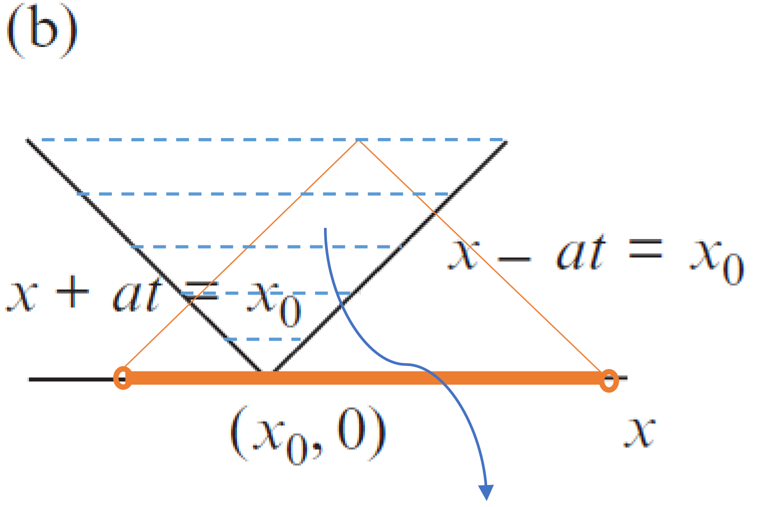
$$u(x,0) = \eta(x) , \quad t > 0$$

The analytic solution is $u(x,t) = \eta(x-at).$

- 1D wave propagates in two directions



The domain of dependence



The domain of influence

The solution $u(x,t)$ at a point (x_0, t_0) depends on the initial conditions only in the interval of $(x_0 - at_0, x_0 + at_0)$.

Solution value $u(x,t), t > 0,$ in the cone formed by the characteristic lines $x + at = x_0$ and $x - at = x_0$ depends on the initial values at $(x_0, 0)$.

5.6.1 An FD Method (CT–CT) for Second-order Wave Equations

$$u_{tt} = a^2 u_{xx}\,, \qquad 0 < x < 1\,,$$

$$\text{IC:} \quad u(x,0) = u_0(x)\,, \quad u_t(x,0) = u_1(x)\,,$$

$$\text{BC:} \quad u(0,t) = g_1(t)\,, \quad u(1,t) = g_2(t)\,.$$

Central finite difference discretization both in time and space (CT-CT):

$$\frac{U_j^{k+1} - 2U_j^k + U_j^{k-1}}{(\Delta t)^2} = a^2 \frac{U_{j-1}^k - 2U_j^k + U_{j+1}^k}{h^2}\,, \qquad (5.26)$$

- Second-order accurate both in time and space $((\Delta t)^2 + h^2)$.
- The CFL constraint for this method is $\Delta t \le \frac{h}{|a|}$.
- The values of $U_j^{-1} \sim u(x_j, -\Delta t)$ is not explicitly defined, how to start the time stepping?

  The IC $u_t(x,0) = u_1(x)$ can be used here! Two methods are as follows:

$u_t(x,0) = u_1(x)$

1. Forward Euler method: $U_j^1 = U_j^0 + \Delta t\, u_1(x_j)$
2. Ghost point method: $U_j^{-1} = U_j^1 - 2\Delta t\, u_1(x_j)$

Substitute into (5.26)

$$U_j^1 = U_j^0 + \Delta t u_1(x_j) + \frac{a^2 \Delta t^2}{2h^2}\left(U_{j-1}^0 - 2U_j^0 + U_{j+1}^0\right)$$

5.6.1.1 The Stability Analysis of the CT-CT scheme

The von Neumann analysis gives

$$\frac{g-2+1/g}{(\Delta t)^2}=a^2\frac{e^{-ih\xi}-2+e^{ih\xi}}{h^2}.$$

When $\mu=|a|\Delta t/h$, using $1-\cos(h\xi)=2\sin^2(h\xi/2)$, this equation becomes

$$g^2-2g+1=\left(-4\mu^2\sin^2\theta\right)g,$$

or

$$g^2-\left(2-4\mu^2\sin^2\theta\right)g+1=0,$$

where $\theta=h\xi/2$, with solution

$$g=1-2\mu^2\sin^2\theta\pm\sqrt{(1-2\mu^2\sin^2\theta)^2-1}.$$

Note that $1-2\mu^2\sin^2\theta \le 1$. If we also have $1-2\mu^2\sin^2\theta < -1$, then one of the roots is

$$g_1 = 1-2\mu^2\sin^2\theta - \sqrt{(1-2\mu^2\sin^2\theta)^2 - 1} < -1$$

For some μ and θ, unstable

so $|g_1| > 1$ for some θ, such that the scheme is unstable.

To have a stable scheme, we require $1-2\mu^2\sin^2\theta \ge -1$, or $\mu^2\sin^2\theta \le 1$, which can be guaranteed if $\mu^2 \le 1$ or $\Delta t \le h/|a|$. This is the CFL condition expected. Under this CFL constraint,

$$|g|^2 = \left(1-2\mu^2\sin^2\theta\right)^2 + \left(1-\left(1-2\mu^2\sin^2\theta\right)^2\right) = 1.$$

If $\mu^2 \le 1$, for any θ, stable

5.8 Finite Difference Methods for Conservation Laws

The canonical form for the 1D conservation law is

$$\mathbf{u}_t + \mathbf{f}(u)_x = 0\,, \tag{5.34}$$

and one famous benchmark problem is Burgers' equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0\,, \tag{5.35}$$

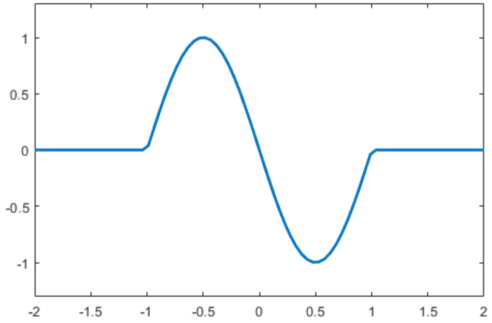
in which $f(u) = u^2/2$. The term $\mathbf{f}(u)$ is often called the flux. This equation can be written in the nonconservative form

$$u_t + uu_x = 0\,, \tag{5.36}$$

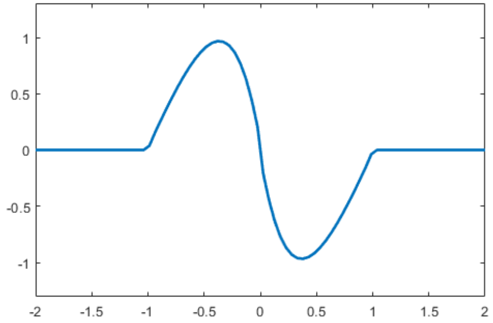
Note for conservation law problems:

The solution likely develops shock(s) where the solution is discontinuous, even if the initial condition is arbitrarily differentiable, *i.e.*, $u_0(x) = \sin x$.

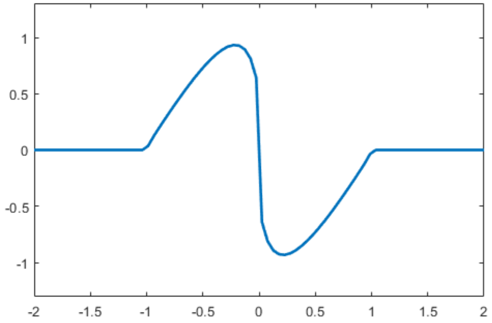
Example: $x \in [-2,2]$, $u_0(x) = \sin(\pi x + 1)$, the periodic condition is applied.



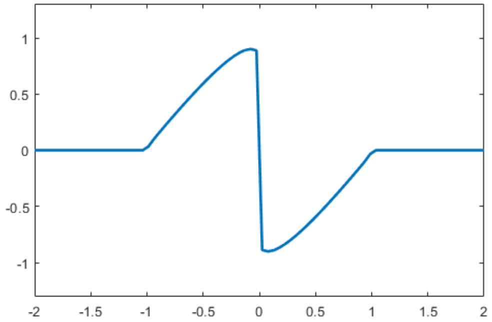
Time step 0 (initial condition)



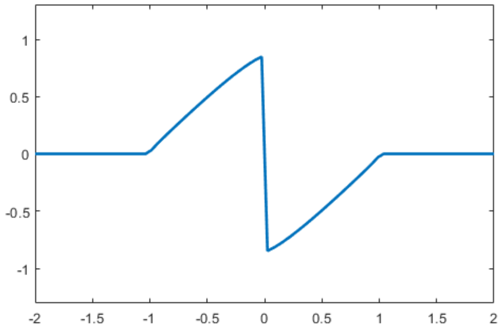
Time step 5



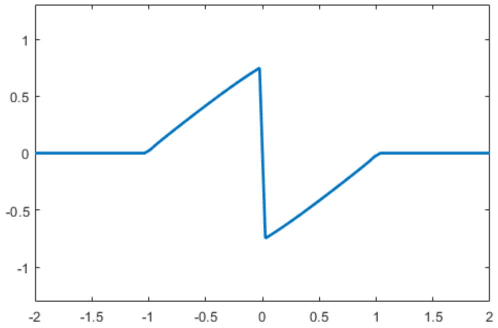
Time step 10



Time step 15



Time step 20



Time step 25

Upwind scheme for the Burgers' equation

(first-order accurate in space and time)

The conservative form:

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + \frac{(U_j^k)^2 - (U_{j-1}^k)^2}{2h} = 0\,, \quad \text{if } U_j^k \geq 0\,,$$

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + \frac{(U_{j+1}^k)^2 - (U_j^k)^2}{2h} = 0\,, \quad \text{if } U_j^k < 0\,.$$

The conservative form is better if shocks develop

The non-conservative form:

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + U_j^k \frac{U_j^k - U_{j-1}^k}{h} = 0\,, \quad \text{if } U_j^k \geq 0\,,$$

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + U_j^k \frac{U_{j+1}^k - U_j^k}{h} = 0\,, \quad \text{if } U_j^k < 0\,,$$

Lax–Wendroff scheme for the Burgers' equation (non-conservative form)

(second-order accurate in space and time)

Review the L-W scheme for linear problem on page 23-25.

Step 1. $$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t}=u_t+\frac{\Delta t}{2}u_{tt}+O((\Delta t)^2)$$

Step 2.
$$\begin{aligned} u_{tt} &= -u_t u_x - u u_{tx} \\ &= u u_x^2 + u(u u_x)_x \\ &= u u_x^2 + u\left(u_x^2 + u u_{xx}\right) \\ &= 2u u_x^2 + u^2 u_{xx}\,, \end{aligned}$$

The modified equation

$$u_t + u u_x = \frac{\Delta t}{2}\left(2u u_x^2 + u^2 u_{xx}\right)$$

Step 3. Apply finite difference for the modified equation, we obtain the L-W scheme:

$$U_j^{k+1} = U_j^k - \Delta t\, U_j^k \frac{U_{j+1}^k - U_{j-1}^k}{2h}$$

$$= + \frac{(\Delta t)^2}{2}\left(2U_j^k\left(\frac{U_{j+1}^k - U_{j-1}^k}{2h}\right)^2 + (U_j^k)^2\frac{U_{j-1}^k - 2U_j^k + U_{j+1}^k}{h^2}\right).$$

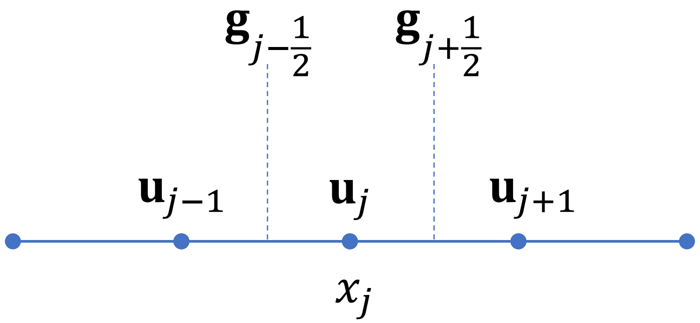
5.8.1 Conservative FD Methods for Conservation Laws

Consider the conservation law

$$\mathbf{u}_t + \mathbf{f}(u)_x = 0\,,$$

and let us seek a numerical scheme of the form

$$\mathbf{u}_j^{k+1} = \mathbf{u}_j^k - \frac{\Delta t}{h}\left(\mathbf{g}_{j+\frac{1}{2}}^k - \mathbf{g}_{j-\frac{1}{2}}^k\right), \tag{5.38}$$



where

$$\mathbf{g}_{j+\frac{1}{2}} = \mathbf{g}\left(\underbrace{\mathbf{u}_{j-p+1}^k, \mathbf{u}_{j-p+2}^k, \ldots, \mathbf{u}_{j+q+1}^k}_{\text{Stencil}}\right)$$

is called the numerical flux, satisfying

$$g(u, u, \ldots, u) = f(u)\,. \tag{5.39}$$

Such a scheme is called conservative. For example, we have $g(u) = u^2/2$ for Burgers' equation.

For a scalar conservation law, how to find g?

Step 1. Integrate the equation with respect to x from $x_{j-\frac{1}{2}}$ to $x_{j+\frac{1}{2}}$, to get

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_t dx = -\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} f(u)_x dx$$

$$= -\left(f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t))\right).$$

Step 2. Integrate the equation above with respect to t from t^k to t^{k+1}, to get

$$\int_{t^k}^{t^{k+1}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_t \, dx \, dt = -\int_{t^k}^{t^{k+1}} \left(f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t))\right) dt,$$

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left(u(x, t^{k+1}) - u(x, t^k)\right) dx = -\int_{t^k}^{t^{k+1}} \left(f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t))\right) dt.$$

Define the average of $u(x,t)$ as

$$\bar{u}_j^k = \frac{1}{h}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,t^k)\,dx\,, \tag{5.40}$$

which is the cell average of $u(x,t)$ over the cell $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ at the time level k. The expression that we derived earlier can therefore be rewritten as

$$\begin{aligned}
\bar{u}_j^{k+1} &= \bar{u}_j^k - \frac{1}{h}\left(\int_{t^k}^{t^{k+1}} f(u(x_{j+\frac{1}{2}},t))dt - \int_{t^k}^{t^{k+1}} f(u(x_{j-\frac{1}{2}},t))dt\right) \\
&= \bar{u}_j^k - \frac{\Delta t}{h}\left(\frac{1}{\Delta t}\int_{t^k}^{t^{k+1}} f(u(x_{j+\frac{1}{2}},t))dt - \frac{1}{\Delta t}\int_{t^k}^{t^{k+1}} f(u(x_{j-\frac{1}{2}},t))dt\right) \\
&= \bar{u}_j^k - \frac{\Delta t}{h}\left(g_{j+\frac{1}{2}} - g_{j+\frac{1}{2}}\right), \quad \text{where } g_{j+\frac{1}{2}} = \frac{1}{\Delta t}\int_{t^k}^{t^{k+1}} f(u(x_{j+\frac{1}{2}},t))dt\,.
\end{aligned}$$

Step 3. Approximate this integral to obtain a (Finite Volume) scheme.

5.8.2 Some Commonly Used Numerical Scheme for Conservation Laws

- Lax–Friedrichs scheme

$$U_j^{k+1} = \frac{1}{2}\left(U_{j+1}^k + U_{j-1}^k\right) - \frac{\Delta t}{2h}\left(f(U_{j+1}^k) - f(U_{j-1}^k)\right); \qquad (5.41)$$

- Lax–Wendroff scheme

$$\begin{aligned} U_j^{k+1} = U_j^k &- \frac{\Delta t}{2h}\left(f(U_{j+1}^k) - f(U_{j-1}^k)\right) \\ &+ \frac{(\Delta t)^2}{2h^2}\left\{A_{j+\frac{1}{2}}\left(f(U_{j+1}^k) - f(U_j^k)\right) - A_{j-\frac{1}{2}}\left(f(U_j^k) - f(U_{j-1}^k)\right)\right\}, \end{aligned} \qquad (5.42)$$

where $A_{j+\frac{1}{2}} = Df(u(x_{j+\frac{1}{2}}, t))$ is the Jacobian matrix of $f(u)$ at $u(x_{j+\frac{1}{2}}, t)$.

A modified version

$$\begin{cases} U_{j+\frac{1}{2}}^{k+\frac{1}{2}} = \frac{1}{2}\left(U_j^k + U_{j+1}^k\right) - \frac{\Delta t}{2h}\left(f(U_{j+1}^k) - f(U_j^k)\right) \\ U_j^{k+1} = U_j^k - \frac{\Delta t}{h}\left(f(U_{j+\frac{1}{2}}^{k+\frac{1}{2}}) - f(U_{j-\frac{1}{2}}^{k+\frac{1}{2}})\right), \end{cases} \qquad (5.43)$$

the Lax–Wendroff–Richtmyer scheme, does not need the Jacobian matrix.

Some comments

- For linear hyperbolic problems, if the initial data is smooth (no discontinuities), it is recommended to use second-order accurate methods such as the Lax–Wendroff method.

- If the initial data has finite discontinuities, called shocks, as second- or high-order methods often lead to oscillations near the discontinuities (Gibbs phenomena)

- For a conservative nonlinear hyperbolic system, shocks may develop in finite time even if the initial data is smooth.

- Explicit methods are preferred for hyperbolic differential equations, usually there is no strict time step constraint as for parabolic problems.