Finite Difference Methods for Hyperbolic PDEs

MATH 3014
Monday & Thursday 14:30-15:45
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For the second-order canonical form

\[ a(x, t)u_{tt} + 2b(x, t)u_{xt} + c(x, t)u_{xx} + \text{lower-order terms} = f(x, t) \]

is hyperbolic if \( b^2 - ac > 0 \) in the entire \( x \) and \( t \) domain.

A few typical model problems involving hyperbolic PDEs are as follows:

- Second-order linear wave equation:
  \[ u_{tt} = au_{xx} \quad , \quad 0 < x < 1 \quad , \]
  \[ u(x, 0) = \eta(x), \quad \frac{\partial u}{\partial t}(x, 0) = \upsilon(x), \quad \text{IC} \quad , \]
  \[ u(0, t) = g_l(t), \quad u(1, t) = g_r(t), \quad \text{BC} \quad . \]
• Advection equation (one-way wave equation):

\[ u_t + au_x = 0, \quad 0 < x < 1, \]
\[ u(x, 0) = \eta(x), \quad \text{IC}, \]
\[ u(0, t) = g_l(t) \quad \text{if} \quad a \geq 0, \quad \text{or} \quad u(1, t) = g_r(t) \quad \text{if} \quad a \leq 0. \]

Here \( g_l \) and \( g_r \) are prescribed boundary conditions from the left and right, respectively.

• Linear first-order hyperbolic system:

\[ u_t = Au_x + f(x, t), \]

where \( u \) and \( f \) are two vectors and \( A \) is a matrix. The system is called hyperbolic if \( A \) is diagonalizable, i.e., if there is a nonsingular matrix \( T \) such that \( A = TDT^{-1} \), and all eigenvalues of \( A \) are real numbers.
• Nonlinear hyperbolic equation or system, notably conservation laws:

\[ u_t + f(u)_x = 0, \quad e.g., \quad \text{Burgers’ equation} \quad u_t + \left( \frac{u^2}{2} \right)_x = 0; \]

\[ u_t + f_x + g_y = 0. \]

For nonlinear hyperbolic PDE, shocks (a discontinuous solution) can develop even if the initial data is smooth.
Some phenomena of shocks

Sound barrier

Dam breaking

Forward-facing step flow
5.1 Characteristics and Boundary Conditions

The exact solution for the one-way wave equation

\[ u_t + au_x = 0, \quad -\infty < x < \infty, \]

\[ u(x, 0) = \eta(x), \quad t > 0 \]

is \( u(x, t) = \eta(x - at) \).

For the finite domain problem

\[ u_t + au_x = 0, \quad 0 < x < 1, \]

\[ u(x, 0) = \eta(x), \quad t > 0, \quad u(0, t) = g(t) \text{ if } a > 0 \]

We consider the method of characteristics in which the solution is constant along the characteristics.
The method of characteristics

For the IVP of ODE
\[
\begin{cases} 
\frac{dx}{dt} = a \\
x(0) = c
\end{cases}
\]

The solution \( x(t, c) = at + c \) is called the characteristic line of the IVP of PDE.

Along the characteristic line \( x = x(t, c) \), \( u = (x(t, c), t) \) satisfies the following ODE
\[
\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt} = \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0,
\]
which means that \( u(x, t) \) is constant along the characteristic line.

Therefore, we have
\[
\begin{align*}
u(x(t, c), t) &= u(x(0, c), 0) = u(c, 0) = \eta(c) \\
u(x, t) &= \eta(x - at).
\end{align*}
\]
The $x - t$ diagram

The $x - u$ diagram
5.2 Finite Difference Schemes for Hyperbolic equations

- Lax–Friedrichs method;
- Upwind scheme;
- Leap-frog method;
- Lax–Wendroff method;
- Crank–Nicolson scheme; and
- Beam–Warming method.
Consider the one-way wave equation \( u_t + au_x = 0 \)

The FW-CT scheme:

\[
U_{j}^{k+1} = U_{j}^{k} - \mu \left( U_{j+1}^{k} - U_{j-1}^{k} \right), \quad \mu = a \Delta t / (2h).
\]

- The local truncation error is \( O(\Delta t + h^2) \)
- The method is unconditionally unstable

The von Neumann stability analysis:

The growth factor \( g(\theta) = 1 - \mu \left( e^{ih\xi} - e^{-ih\xi} \right) \)

\[
= 1 - \mu 2 i \sin(h\xi), \text{ where } \theta = h\xi, \mu = a \Delta t / (2h).
\]

we have

\[
|g(\theta)|^2 = 1 + 4\mu^2 \sin^2(h\xi) \geq 1.
\]
The Lax–Friedrichs scheme: average $U_j^k$ using $U_{j-1}^k$ and $U_{j+1}^k$ to get

\[ U_j^{k+1} = \frac{1}{2} \left( U_{j-1}^k + U_{j+1}^k \right) - \mu \left( U_{j+1}^k - U_{j-1}^k \right). \]

- The local truncation error is $O(\Delta t + h)$ if $\Delta t \approx h$.
- The method is conditionally stable

The growth factor:

\[ g(\theta) = \frac{1}{2} (e^{ih\xi} + e^{-ih\xi}) + \mu (e^{ih\xi} - e^{-ih\xi}) = \cos(h\xi) - 2\mu \sin(h\xi)i \]

\[ |g(\theta)|^2 = \cos^2(h\xi) + 4\mu^2 \sin^2(h\xi) = 1 - \sin^2(h\xi) + 4\mu^2 \sin^2(h\xi) = 1 - (1 - 4\mu^2) \sin^2(h\xi), \]

Therefore, if $\Delta t \leq h/|a|$ \[ 1 - 4\mu^2 \geq 0 \] \[ |g(\theta)| \leq 1 \]
clear; close all

a = 0; b=1; tfinal = 0.5; m=20;
h = (b-a)/m;
k = h; mu = k/h;
t = 0;
n = fix(tfinal/k);
y1 = zeros(m+1,1); y2=y1; x=y1;
% Initial condition
for i=1:m+1,
x(i) = a + (i-1)*h;
y1(i) = uexact(t,x(i)); % current
ty2(i) = 0; % next level
end

t = 0;
for j=1:n,
y1(1)=bc(t);
y2(1)=bc(t+k); % Physical boundary condition
for i=2:m
    % FW-CT scheme
    % y2(i) = y1(i) - mu*(y1(i+1)-y1(i-1))/2;
    % Lax-Friedrichs scheme
    y2(i) = 0.5*(y1(i+1)+y1(i-1)) - mu*(y1(i+1)-y1(i-1))/2;
end
i = m+1;
% Numerical boundary condition
y2(i) = y1(i) - mu*(y1(i)-y1(i-1) );
t = t + k;
y1 = y2;
plot(x,y2); pause(0.5)
end
**Problem**: the 1D advection equation

\[ u_t + u_x = 0 \text{ in the domain } 0 < x < 1. \]

The initial condition is \( u(x, 0) = u_0(x) = \begin{cases} 0 & \text{if } 0 < x < 1/2, \\ 1 & \text{if } 1/2 \leq x < 1. \end{cases} \)

The boundary condition is \( u(0, t) = \sin t. \)

The analytic solution is \( u(x, t) = \begin{cases} u_0(x - t) & \text{if } 0 < t < x < 1, \\ \sin(t - x) & \text{if } 0 < x < t < 1. \end{cases} \)

\( t - x \) is the time period that the \( \sin \) function experiences.
Solutions at different time steps obtained using the Lax–Friedrichs scheme when $\Delta t = h$.
Solutions at different time steps obtained using the Lax–Friedrichs scheme when $\Delta t = 1.5h$ (Blow up)
The Upwind scheme:

The upwind scheme for \( u_t + au_x = 0 \) is

\[
\frac{U_j^{k+1} - U_j^k}{\Delta t} = \begin{cases} 
-\frac{a}{h} \left( U_j^k - U_{j-1}^k \right) & \text{if } a \geq 0, \\
-\frac{a}{h} \left( U_{j+1}^k - U_j^k \right) & \text{if } a < 0,
\end{cases}
\]

• The scheme is first-order accurate in time and in space
• The method is \textit{conditionally stable}

The growth factor for the case when \( a \geq 0 \) is:

\[
g(\theta) = 1 - \mu \left( 1 - e^{-ih\xi} \right) = 1 - \mu(1 - \cos(h\xi)) - i\mu \sin(h\xi)
\]

\[
|g(\theta)|^2 = (1 - \mu + \mu \cos(h\xi))^2 + \mu^2 \sin^2(h\xi) = (1 - \mu)^2 + 2(1 - \mu)\mu \cos(h\xi) + \mu^2 = 1 - 2(1 - \mu)\mu(1 - \cos(h\xi)),
\]

so if \( 1 - \mu \geq 0 \) (i.e., \( \mu \leq 1 \)) or \( \Delta t \leq h/a \) we have \( |g(\theta)| \leq 1. \)
clear; close all

a = 0; b=1; tfinal = 0.5; m = 20;

aa = 1; % The coefficient
h = (b-a)/m; k = h/abs(aa);
mu = aa*k/h; % Set mesh and time step.

t = 0; n = fix(tfinal/k);
y1 = zeros(m+1,1); y2=y1; x=y1;

figure(1);
%axis([-0.1 1.1 -0.1 1.1]);
for i=1:m+1,
    x(i) = a + (i-1)*h;
y1(i) = uexact(t,x(i)); % Initial data
    y2(i) = 0;
end

% Time marching
for j=1:n,
y1(1)=bc(t);
y2(1)=bc(t+k);
for i=2:m+1
    y2(i) = y1(i) - mu*(y1(i)-y1(i-1 ));
end
t = t + k;
y1 = y2;
plot(x,y2); pause(0.5);
End

% Define exact solution for comparison
u_e = zeros(m+1,1);
for i=1:m+1
    u_e(i) = uexact(t,x(i));
end

max(abs(u_e-y2))
plot(x,y2,'o',x,u_e)
Solutions at different time steps obtained using the Upwind scheme when $\Delta t = 0.5h$ (Smooth out effect)
The Leap-frog Scheme:

The leap-frog scheme for $u_t + au_x = 0$ is

$$\frac{U_j^{k+1} - U_j^{k-1}}{2\Delta t} + \frac{a}{2h} \left( U_{j+1}^k - U_{j-1}^k \right) = 0,$$

or

$$U_j^{k+1} = U_j^{k-1} - \frac{a}{h} \left( U_{j+1}^k - U_{j-1}^k \right),$$

- The discretization is second-order in time and in space.
- The method is **conditionally stable**, CFL condition: $\Delta t < \frac{h}{|a|}$.
- It requires an Numerical Boundary Condition at one end.
- It needs $U_j^1$ to get started, we can use the upwind or other scheme to obtain $U_j^1$. 
The von Neumann analysis for the leap scheme

Substituting

\[ U_j^k = e^{ij\xi}, \quad U_j^{k+1} = g(\xi)e^{ij\xi}, \quad U_j^{k-1} = \frac{1}{g(\xi)} e^{ij\xi} \]

into the leap-frog scheme, we get

\[ g^2 + \mu(e^{ih\xi} - e^{-ih\xi})g - 1 = 0, \]

or

\[ g^2 + 2\mu i \sin(h\xi) g - 1 = 0, \]

with solution

\[ g_\pm = -i\mu \sin(h\xi) \pm \sqrt{1 - \mu^2 \sin^2(h\xi)}. \quad (5.10) \]

We distinguish three different cases.

1. If \(|\mu| > 1\), then there are \(\xi\) such that at least one of \(|g_-| > 1\) or \(|g_+| > 1\) holds, so the scheme is unstable!
2. If \(|\mu| < 1\), then \(1 - \mu^2 \sin^2(h\xi) \geq 0\) such that

\[ |g_\pm|^2 = \mu^2 \sin^2(h\xi) + 1 - \mu^2 \sin^2(h\xi) = 1. \]
3. If $|\mu| = 1$, we still have $|g_\pm| = 1$, but we can find $\xi$ such that $\mu \sin(h\xi) = 1$ and $g_+ = g_- = -i$, i.e., $-i$ is a double root of the characteristic polynomial. The solution of the finite difference equation therefore has the form

$$U_j^k = C_1 (-i)^k + C_2 k (-i)^k,$$

where the possibly complex numbers $C_1$ and $C_2$ are determined from the initial conditions. Thus there are solutions such that $\|U^k\| \simeq k$ which are unstable (slow growing).
5.3 The Modified PDE and Numerical Diffusion/Dispersion

A modified PDE is the PDE that a finite difference equation \textit{satisfies exactly} at grid points.

Take the upwind method for the advection equation \( u_t + au_x = 0 \) with \( a > 0 \) for example

\[
\frac{U_j^{k+1} - U_j^k}{\Delta t} + \frac{a}{h} \left( U_j^k - U_{j-1}^k \right) = 0. 
\]

The derivation of a modified PDE is similar to computing the local truncation error.

Insert \( v(x, t) \) in to the finite difference equation to \textit{derive a PDE that} \( v(x, t) \) \textit{satisfies exactly}.

\[
\frac{v(x, t + \Delta t) - v(x, t)}{\Delta t} + \frac{a}{h} \left( v(x, t) - v(x - h, t) \right) = 0. 
\]
\[
\frac{v(x, t + \Delta t) - v(x, t)}{\Delta t} + \frac{a}{h} (v(x, t) - v(x - h, t)) = 0.
\]

Expanding the terms in Taylor series about \((x, t)\) and simplifying yields

\[
v_t + \frac{1}{2} \Delta t v_{tt} + \cdots + a \left( v_x - \frac{1}{2} h v_{xx} + \frac{1}{6} h^2 v_{xxx} + \cdots \right) = 0,
\]

which can be rewritten as

\[
v_t + av_x = \frac{1}{2} (ah v_{xx} - \Delta t v_{tt}) - \frac{1}{6} \left( ah^2 v_{xxx} + (\Delta t)^2 v_{tt} \right) + \cdots,
\]

which is the PDE that \(v\) satisfies. Consequently,

\[
\frac{\partial}{\partial t} \text{ on both sides} \quad v_{tt} = -av_{xt} + \frac{1}{2} (ah v_{xx} - \Delta t v_{tt})
\]

\[
= -av_{xt} + O(\Delta t, h)
\]

\[
= -a \frac{\partial}{\partial x} \left( -av_x + O(\Delta t, h) \right),
\]

so the leading modified PDE is

\[
v_t + av_x = \frac{1}{2} ah \left( 1 - \frac{a \Delta t}{h} \right) v_{xx}.
\] (5.11)

Advection–diffusion equation
The original PDE
\[ u_t + au_x = 0, \ a > 0 \]

A first-order accurate approximation to the true solution of the original PDE

The upwind scheme
\[ \frac{U_j^{k+1} - U_j^k}{\Delta t} + \frac{a}{h} \left( U_j^k - U_j^{k-1} \right) = 0 \]

Satisfies exactly

The modified PDE
\[ v_t + av_x = \frac{1}{2} (ahv_{xx} - \Delta tv_{tt}) - \frac{1}{6} \left( ah^2 v_{xxx} + (\Delta t)^2 v_{tt} \right) + \cdots \]

The high order terms are \( O(h^2 + \Delta t^2) + O(\Delta t^2 h) \)

A second-order accurate approximation to the true solution of the leading modified PDE

The leading modified PDE
\[ v_t + av_x = \frac{1}{2} ah \left( 1 - \frac{a\Delta t}{h} \right) v_{xx} \]
The modified equation tells some features of the scheme:

- The computed solution smooths out discontinuities because of the diffusion term \( \frac{1}{2} ah \left( 1 - \frac{a \Delta t}{h} \right) v_{xx} \).

- We have second-order accuracy to \( u_t + au_x = 0 \) if \( a \) is a constant and \( \Delta t = h/a \).

- We can add the correction term to offset the leading error term to render a higher-order accurate method, but the stability needs to be checked. For instance, we can modify the upwind scheme to get a second-order scheme when \( \Delta t \simeq h \):

\[
\frac{U_{j}^{k+1} - U_{j}^{k}}{\Delta t} + a \frac{U_{j}^{k} - U_{j-1}^{k}}{h} = \frac{1}{2} ah \left( 1 - \frac{a \Delta t}{h} \right) \frac{U_{j-1}^{k} - 2U_{j}^{k} + U_{j+1}^{k}}{h^2}.
\]

This approximates \( u_{xx} \) with \( O(h^2) \), therefore the RHS of (5.11) can be cancelled with \( O(h^3) \).
Why some schemes are unstable?

--- check the modified equation

The PDE:

\[ u_t + au_x = 0 \]

The FW-CT scheme:

\[
\frac{U_j^{k+1} - U_j^k}{\Delta t} + a \frac{U_{j+1}^k - U_{j-1}^k}{2h} = 0
\]

\[
\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = u_t + \frac{\Delta t}{2} u_{tt} + O((\Delta t)^2)
\]

\[
= u_t + \frac{1}{2} a^2 (\Delta t) u_{xx} + O((\Delta t)^2).
\]

The leading term of the modified PDE for the FW-CT scheme:

\[
v_t + av_x = - \frac{a^2 \Delta t}{2} v_{xx}
\]

The sign is negative here! Similar to the backward heat equation that is dynamically unstable.
5.4 The Lax–Wendroff Scheme

Note that for the time discretization of the PDE $u_t + au_x = 0$:

$$\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = u_t + \frac{\Delta t}{2} u_{tt} + O((\Delta t)^2)$$

$$= u_t + \frac{1}{2} a^2(\Delta t) u_{xx} + O((\Delta t)^2).$$

Recall that $T(x) = \frac{u(x - h) - 2u(x) + u(x + h)}{h^2} - u_{xx}(x) = \frac{h^2}{12} u^{(4)}(x) + \cdots = O(h^2)$

Hence $\frac{1}{2} a^2(\Delta t) u_{xx}$ can be cancelled by using the central finite difference with 2\textsuperscript{nd} order accuracy.

The Lax–Wendroff scheme:

$$\frac{U_{j}^{k+1} - U_{j}^{k}}{\Delta t} + a \frac{U_{j+1}^{k} - U_{j-1}^{k}}{2h} = \frac{1}{2} \frac{a^2 \Delta t}{h^2} \left( U_{j-1}^{k} - 2U_{j}^{k} + U_{j+1}^{k} \right), \quad (5.14)$$

The derivation of the L-W scheme is easier than the derivation of the modified upwind second-order scheme on Page 21, because the central difference for the first-order term $au_x$ already gives a high order truncation error, here we only do the Taylor expansion w.r.t $t$. 
To derive the Lax-Wendroff scheme

1. Do the Taylor expansion only with respect to \( t \).

2. Make use of the original PDE to transform \( u_{tt} \) to a term involving the derivatives w.r.t \( x \) (i.e., \( u_{xx} \)), the resulting formulation is called the modified PDE.

3. Apply finite difference for the term involving the derivatives w.r.t \( x \) (Spatial discretization).
The Lax–Wendroff scheme is second-order accurate both in time and space.

The local truncation error of the Lax–Wendroff scheme:

\[
T(x, t) = \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} - \frac{a(u(x + h, t) - u(x - h, t))}{2h} \\
- \frac{a^2 \Delta t (u(x - h, t) - 2u(x, t) + u(x + h, t))}{2h^2}
\]

\[
= u_t + \frac{\Delta t}{2} u_{tt} - a u_x - \frac{a^2 \Delta t}{2} u_{xx} + O((\Delta t)^2 + h^2)
\]

\[
= O((\Delta t)^2 + h^2),
\]

\[
u_t = -au_x
\]

\[
u_{tt} = -au_{xt} = -a \frac{\partial}{\partial x} u_t = a^2 u_{xx}
\]
The CFL condition for the Lax–Wendroff scheme

The von Neumann stability analysis

\[ g(\theta) = 1 - \frac{\mu}{2} (e^{ih\xi} - e^{-ih\xi}) + \frac{\mu^2}{2} (e^{-ih\xi} - 2 + e^{ih\xi}) \]

\[ = 1 - \mu i \sin \theta - 2\mu^2 \sin^2(\theta/2), \]

where again \( \theta = h\xi \), so

\[ |g(\theta)|^2 = \left( 1 - 2\mu^2 \sin^2 \frac{\theta}{2} \right)^2 + \mu^2 \sin^2 \theta \]

\[ = 1 - 4\mu^2 \sin^2 \frac{\theta}{2} + 4\mu^4 \sin^4 \frac{\theta}{2} + 4\mu^2 \sin^2 \frac{\theta}{2} \left( 1 - \sin^2 \frac{\theta}{2} \right) \]

\[ = 1 - 4\mu^2 \left( 1 - \mu^2 \right) \sin^4 \frac{\theta}{2} \]

\[ \leq 1 - 4\mu^2 \left( 1 - \mu^2 \right) \quad \text{If this is positive, then stable} \]

We conclude \(|g(\theta)| \leq 1\) if \( \mu \leq 1 \), i.e., \( \Delta t \leq h/|a| \). If \( \Delta t > h/|a| \), there are \( \xi \) such that \(|g(\theta)| > 1\) so the scheme is unstable.
Advantages of the Lax–Wendroff scheme:

- Second-order accurate both in time and space
- Conditionally stable ($\Delta t \leq \frac{h}{|a|}$)

Disadvantages of the Lax–Wendroff scheme:

- Leads to a dispersive modified PDE
  \[ v_t + av_x = -\frac{1}{6}ah^2 \left( 1 - \left( \frac{a\Delta t}{h} \right)^2 \right) v_{xxx} \]

- The numerical result can be expected to develop a train of oscillations behind the discontinuity
The Lax–Wendroff method for $u_t + au_x = 0$:

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} + a \frac{U_{j+1}^k - U_{j-1}^k}{2h} = \frac{1}{2} \frac{a^2 \Delta t}{h^2} \left( U_{j-1}^k - 2U_j^k + U_{j+1}^k \right), \quad (5.14)$$

When $a > 0$:

When $a < 0$:

The Beam–Warming method for $u_t + au_x = 0$ for $a > 0$ is

$$U_j^{k+1} = U_j^k - \frac{a \Delta t}{2h} \left( 3U_j^k - 4U_{j-1}^k + U_{j-2}^k \right) + \frac{(a \Delta t)^2}{2h^2} \left( U_j^k - 2U_{j-1}^k + U_{j-2}^k \right), \quad (5.18)$$
5.4 The Beam–Warming scheme

• The Beam–Warming method is second-order accurate in time and space if $\Delta t \approx h$.

Recall the one-sided finite difference formulas

\[ u'(x) = \frac{3u(x) - 4u(x - h) + u(x - 2h)}{2h} + O(h^2), \]

\[ u''(x) = \frac{u(x) - 2u(x - h) + u(x - 2h)}{h^2} + O(h). \]

• The CFL constraint is $0 < \Delta t \leq \frac{2h}{|a|}$.

• For this method, we do not require an Numerical Boundary Condition (NBC) at $x = 1$, but we need a scheme to compute the solution $U'_1$. 
5.4.2 The Crank–Nicolson Scheme

\[
\frac{U_j^{k+1} - U_j^k}{\Delta t} + a \frac{U_{j+1}^k - U_{j-1}^k + U_{j+1}^{k+1} - U_{j-1}^{k+1}}{4h} = f_j^{k+\frac{1}{2}}, \quad (5.21)
\]

- Second-order accurate in time and in space.
- Unconditionally stable.
- An NBC is needed at \( x = 1 \) for case \( a > 0 \).
- This method is effective for the 1D problem, since it is easy to solve the resulting tridiagonal system of equations.
- For 2D and 3D, use Alternating Direction Implicit (ADI) Method.
5.5 Numerical Boundary Condition

For the one-way wave equation $u_t + au_x = 0$, we need a numerical boundary condition (NBC) at one end when we use any of the “central type” FDM, i.e., the Lax–Friedrichs, Lax–Wendroff, or leapfrog schemes.

- We have one (and only one) physical boundary condition at one end.
- For “upwind type” FDM, we don’t need NBC, i.e., the Upwind scheme, the Beam-Warming scheme.

First-order approximation: $U_{M}^{k+1} = U_{M-1}^{k+1}$.

Second-order approximation: $U_{M}^{k+1} = U_{M-2}^{k+1} \frac{x_M - x_{M-1}}{x_{M-1} - x_M} + U_{M-1}^{k+1} \frac{x_M - x_{M-2}}{x_{M-2} - x_{M-1}}$.

If a uniform grid is used

$$U_{M}^{k+1} = -U_{M-2}^{k+1} + 2U_{M-1}^{k+1}.$$
5.6 Finite Difference Methods for Second-order Linear Hyperbolic PDEs

Modeling 1D sound wave propagates in two directions

\[ u_{tt} = a^2 u_{xx}, \quad \text{where } a > 0 \text{ is the wave speed.} \]

Find the analytic solution by changing variables:

Let \[
\begin{align*}
\xi &= x - at \\
\eta &= x + at
\end{align*}
\]

or \[
\begin{align*}
x &= \frac{\xi + \eta}{2} \\
t &= \frac{\eta - \xi}{2a}
\end{align*}
\]

Using the chain-rule

\[
\begin{align*}
u_t &= -au_\xi + au_\eta, \\
u_{tt} &= a^2 u_{\xi\xi} - 2a^2 u_{\xi\eta} + a^2 u_{\eta\eta}, \\
u_x &= u_\xi + u_\eta, \\
u_{xx} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.
\end{align*}
\]

\[
\begin{align*}
u_{\xi\eta} a^2 - 2a^2 u_{\xi\eta} + a^2 u_{\eta\eta} = a^2 (u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}), \\
4a^2 u_{\xi\eta} = 0
\end{align*}
\]
Find the analytic solution as follows:

\[ u_{tt} = a^2 u_{xx}, \]

Changing of variables

\[ 4a^2 u_{\xi\eta} = 0 \]

\[ u \text{ must not contain any mix terms that depend on both } \xi \text{ and } \eta, \text{ otherwise RHS } \neq 0 \]

\[ u(x, t) = F(\xi) + G(\eta), \]

where \( F(\xi) \) and \( G(\eta) \) are two differential functions of one variable.

\[ u(x, t) = F(x - at) + G(x + at) \]

The two functions \( F, G \) are determined by initial and boundary conditions.
Example: the Cauchy problem

- 1D wave propagates in **two** directions

\[ u_{tt} = d^2 u_{xx}, \quad -\infty < x < \infty, \]
\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = g(x), \]

The analytic solution is called the **D’Alembert’s formula**, as

\[ u(x, t) = \frac{1}{2} (u_0(x - at) + u_0(x + at)) + \frac{1}{2a} \int_{x-at}^{x+at} g(s)ds. \]

- 1D wave propagates in **one** direction

\[ u_t + au_x = 0, \quad -\infty < x < \infty, \]
\[ u(x, 0) = \eta(x), \quad t > 0 \]

The analytic solution is \[ u(x, t) = \eta(x - at). \]
• 1D wave propagates in two directions

![Diagram (a)](image1)

The domain of dependence

The solution \( u(x, t) \) at a point \((x_0, t_0)\) depends on the initial conditions only in the interval of \((x_0 - at_0, x_0 + at_0)\).

![Diagram (b)](image2)

The domain of influence

Solution value \( u(x, t), t > 0 \), in the cone formed by the characteristic lines \( x + at = x_0 \) and \( x - at = x_0 \) depends on the initial values at \((x_0, 0)\).
5.6.1 An FD Method (CT–CT) for Second-order Wave Equations

\[ u_{tt} = a^2 u_{xx}, \quad 0 < x < 1, \]

**IC:** \( u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \)

**BC:** \( u(0, t) = g_1(t), \quad u(1, t) = g_2(t). \)

Central finite difference discretization both in time and space (CT-CT):

\[
\frac{U_j^{k+1} - 2U_j^k + U_j^{k-1}}{(\Delta t)^2} = a^2 \frac{U_{j-1}^k - 2U_j^k + U_{j+1}^k}{h^2}, \tag{5.26}
\]

- Second-order accurate both in time and space \(((\Delta t)^2 + h^2)\).
- The CFL constraint for this method is \( \Delta t \leq \frac{h}{|a|} \).
- The values of \( U_j^{-1} \sim u(x_j, -\Delta t) \) is not explicitly defined, how to start the time stepping?
  The IC \( u_t(x, 0) = u_1(x) \) can be used here! Two methods are as follows:

\[
\begin{align*}
1. \text{Forward Euler method:} & \quad U_j^1 = U_j^0 + \Delta t u_1(x_j) \\
2. \text{Ghost point method:} & \quad U_j^{-1} = U_j^1 - 2\Delta t u_1(x_j) \\
& \quad U_j^1 = U_j^0 + \Delta t u_1(x_j) + \frac{a^2 \Delta t^2}{2h^2} (U_{j-1}^0 - 2U_j^0 + U_{j+1}^0)
\end{align*}
\]

Substitute into (5.26)
5.6.1.1 The Stability Analysis of the CT-CT scheme

The von Neumann analysis gives

\[ \frac{g - 2 + \frac{1}{g}}{(\Delta t)^2} = a^2 \frac{e^{-i\mu} - 2 + e^{i\mu}}{h^2}. \]

When \( \mu = |a| \Delta t / h \), using \( 1 - \cos(h\xi) = 2\sin^2(h\xi/2) \), this equation becomes

\[ g^2 - 2g + 1 = \left(-4\mu^2 \sin^2 \theta\right) g, \]

or

\[ g^2 - \left(2 - 4\mu^2 \sin^2 \theta\right) g + 1 = 0, \]

where \( \theta = h\xi/2 \), with solution

\[ g = 1 - 2\mu^2 \sin^2 \theta \pm \sqrt{(1 - 2\mu^2 \sin^2 \theta)^2 - 1}. \]
Note that \( 1 - 2\mu^2 \sin^2 \theta \leq 1 \). If we also have \( 1 - 2\mu^2 \sin^2 \theta < -1 \), then one of the roots is

\[
g_1 = 1 - 2\mu^2 \sin^2 \theta - \sqrt{(1 - 2\mu^2 \sin^2 \theta)^2 - 1} < -1
\]

so \(|g_1| > 1\) for some \( \theta \), such that the scheme is unstable.

To have a stable scheme, we require \( 1 - 2\mu^2 \sin^2 \theta \geq -1 \), or \( \mu^2 \sin^2 \theta \leq 1 \), which can be guaranteed if \( \mu^2 \leq 1 \) or \( \Delta t \leq h/|a| \). This is the CFL condition expected. Under this CFL constraint,

\[
|g|^2 = \left(1 - 2\mu^2 \sin^2 \theta\right)^2 + \left(1 - \left(1 - 2\mu^2 \sin^2 \theta\right)^2\right) = 1.
\]

If \( \mu^2 \leq 1 \), for any \( \theta \), stable.
The canonical form for the 1D conservation law is

$$u_t + f(u)_x = 0,$$  \hspace{1cm} (5.34)

and one famous benchmark problem is Burgers’ equation

$$u_t + \left( \frac{u^2}{2} \right)_x = 0,$$ \hspace{1cm} (5.35)

in which $f(u) = u^2/2$. The term $f(u)$ is often called the flux. This equation can be written in the nonconservative form

$$u_t + uu_x = 0,$$ \hspace{1cm} (5.36)
Note for conservation law problems:
The solution likely develops shock(s) where the solution is discontinuous, even if the initial condition is arbitrarily differentiable, i.e., \( u_0(x) = \sin x \).

Example: \( x \in [-2,2], u_0(x) = \sin(\pi x + 1) \), the periodic condition is applied.
Upwind scheme for the Burgers’ equation
(first-order accurate in space and time)

The conservative form:

\[
\frac{U_j^{k+1} - U_j^k}{\Delta t} + \frac{(U_j^k)^2 - (U_{j-1}^k)^2}{2h} = 0, \quad \text{if } U_j^k \geq 0, \\
\frac{U_j^{k+1} - U_j^k}{\Delta t} + \frac{(U_{j+1}^k)^2 - (U_j^k)^2}{2h} = 0, \quad \text{if } U_j^k < 0.
\]

The non-conservative form:

\[
\frac{U_j^{k+1} - U_j^k}{\Delta t} + U_j^k \frac{U_j^k - U_{j-1}^k}{h} = 0, \quad \text{if } U_j^k \geq 0, \\
\frac{U_j^{k+1} - U_j^k}{\Delta t} + U_j^k \frac{U_{j+1}^k - U_j^k}{h} = 0, \quad \text{if } U_j^k < 0.
\]
Lax–Wendroff scheme for the Burgers’ equation (non-conservative form) (second-order accurate in space and time)

Review the L-W scheme for linear problem on page 23-25.

Step 1. \[
\frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = u_t + \frac{\Delta t}{2} u_{tt} + O((\Delta t)^2)
\]

Step 2. \[
u_{tt} = -u_t u_x - u u_{tx}
\]
\[
= uu_x^2 + u(uu_x)_x
\]
\[
= uu_x^2 + u \left( u_x^2 + uu_{xx} \right)
\]
\[
= 2uu_x^2 + u^2 u_{xx},
\]

The modified equation

\[
u_t + uu_x = \frac{\Delta t}{2} \left( 2uu_x^2 + u^2 u_{xx} \right)
\]

Step 3. Apply finite difference for the modified equation, we obtain the L-W scheme:

\[
U_{j+1}^{k+1} = U_j^{k+1} - \Delta t U_j^k \frac{U_{j+1}^k - U_{j-1}^k}{2h}
\]

\[
= + \frac{(\Delta t)^2}{2} \left( 2U_j^k \left( \frac{U_{j+1}^k - U_{j-1}^k}{2h} \right)^2 + \left( U_j^k \right)^2 \frac{U_{j-1}^k - 2U_j^k + U_{j+1}^k}{h^2} \right).
\]
5.8.1 Conservative FD Methods for Conservation Laws

Consider the conservation law

$$u_t + f(u)_x = 0,$$

and let us seek a numerical scheme of the form

$$u_{j}^{k+1} = u_{j}^{k} - \frac{\Delta t}{h} \left( g_{j+\frac{1}{2}}^{k} - g_{j-\frac{1}{2}}^{k} \right),$$  \hspace{1cm} (5.38)$$

where

$$g_{j+\frac{1}{2}} = g \left( u_{j-p+1}^{k}, u_{j-p+2}^{k}, \ldots, u_{j+q+1}^{k} \right)$$

is called the numerical flux, satisfying

$$g(u, u, \ldots, u) = f(u).$$  \hspace{1cm} (5.39)$$

Such a scheme is called conservative. For example, we have $g(u) = u^2/2$ for Burgers’ equation.
For a scalar conservation law, how to find g?

Step 1. Integrate the equation with respect to $x$ from $x_{j-\frac{1}{2}}$ to $x_{j+\frac{1}{2}}$, to get

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_t dx = - \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} f(u)_x dx$$

$$= - \left( f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t)) \right).$$

Step 2. Integrate the equation above with respect to $t$ from $t^k$ to $t^{k+1}$, to get

$$\int_{t^k}^{t^{k+1}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_t dx \, dt = - \int_{t^k}^{t^{k+1}} \left( f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t)) \right) dt,$$

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left( u(x, t^{k+1}) - u(x, t^k) \right) dx = - \int_{t^k}^{t^{k+1}} \left( f(u(x_{j+\frac{1}{2}}, t)) - f(u(x_{j-\frac{1}{2}}, t)) \right) dt.$$
Define the average of $u(x, t)$ as

$$\bar{u}_j^k = \frac{1}{h} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x, t^k) \, dx,$$  \hspace{1cm} (5.40)$$

which is the \textbf{cell average} of $u(x, t)$ over the cell $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ at the time level $k$. The expression that we derived earlier can therefore be rewritten as

$$\bar{u}_j^{k+1} = \bar{u}_j^k - \frac{1}{h} \left( \int_{t^k}^{t^{k+1}} f(u(x_{j+\frac{1}{2}}, t)) \, dt - \int_{t^k}^{t^{k+1}} f(u(x_{j-\frac{1}{2}}, t)) \, dt \right)$$

$$= \bar{u}_j^k - \frac{\Delta t}{h} \left( \frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} f(u(x_{j+\frac{1}{2}}, t)) \, dt - \frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} f(u(x_{j-\frac{1}{2}}, t)) \, dt \right)$$

$$= \bar{u}_j^k - \frac{\Delta t}{h} \left( g_{j+\frac{1}{2}} - g_{j+\frac{1}{2}} \right), \quad \text{where} \quad g_{j+\frac{1}{2}} = \frac{1}{\Delta t} \int_{t^k}^{t^{k+1}} f(u(x_{j+\frac{1}{2}}, t)) \, dt.$$

Step 3. Approximate this integral to obtain a (\textbf{Finite Volume}) scheme.
5.8.2 Some Commonly Used Numerical Scheme for Conservation Laws

- Lax–Friedrichs scheme

\[
U_{j}^{k+1} = \frac{1}{2} \left( U_{j+1}^{k} + U_{j-1}^{k} \right) - \frac{\Delta t}{2h} \left( f(U_{j+1}^{k}) - f(U_{j-1}^{k}) \right);
\]  \hspace{1cm} (5.41)

- Lax–Wendroff scheme

\[
U_{j}^{k+1} = U_{j}^{k} - \frac{\Delta t}{2h} \left( f(U_{j+1}^{k}) - f(U_{j-1}^{k}) \right) + \frac{(\Delta t)^2}{2h^2} \left\{ A_{j+\frac{1}{2}} \left( f(U_{j+1}^{k}) - f(U_{j}^{k}) \right) - A_{j-\frac{1}{2}} \left( f(U_{j}^{k}) - f(U_{j-1}^{k}) \right) \right\},
\]  \hspace{1cm} (5.42)

where \( A_{j+\frac{1}{2}} = Df(u(x_{j+\frac{1}{2}}, t)) \) is the Jacobian matrix of \( f(u) \) at \( u(x_{j+\frac{1}{2}}, t) \).

A modified version

\[
\begin{cases}
U_{j+\frac{1}{2}}^{k+\frac{1}{2}} = \frac{1}{2} \left( U_{j}^{k+1} + U_{j+1}^{k} \right) - \frac{\Delta t}{2h} \left( f(U_{j+\frac{1}{2}}^{k+\frac{1}{2}}) - f(U_{j+\frac{1}{2}}^{k+1}) \right), \\
U_{j}^{k+1} = U_{j}^{k} - \frac{\Delta t}{h} \left( f(U_{j+\frac{1}{2}}^{k+\frac{1}{2}}) - f(U_{j-\frac{1}{2}}^{k+\frac{1}{2}}) \right),
\end{cases}
\]  \hspace{1cm} (5.43)

the Lax–Wendroff–Richtmyer scheme, does not need the Jacobian matrix.
Some comments

- For linear hyperbolic problems, if the initial data is smooth (no discontinuities), it is recommended to use second-order accurate methods such as the Lax–Wendroff method.

- If the initial data has finite discontinuities, called shocks, as second- or high-order methods often lead to oscillations near the discontinuities (Gibbs phenomena)

- For a conservative nonlinear hyperbolic system, shocks may develop in finite time even if the initial data is smooth.

- Explicit methods are preferred for hyperbolic differential equations, usually there is no strict time step constraint as for parabolic problems.