Deep and Low-Rank Quaternion Priors for Color Image Processing

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Abstract—Due to the physical nature of color images, color image processing such as denoising and inpainting has shown extensive and versatile possibilities over grayscale image processing. The monochromatic and the concatenation model have been widely used to process color images by processing each color channel independently or concatenating three color channels as one unified one and then used existing grayscale image processing methods directly without specific operations. These above schemes, however, have some limitations: (1) they would destroy the inherent correlation among three color channels since they cannot represent color images holistically; (2) they usually focus on one specific handcrafted prior such as smoothness, low-rankness, or even deep prior and thus failing to fuse deep and handcrafted priors of color images flexibly. To conquer these limitations, we propose one unified model to integrate deep prior and low-rank quaternion prior (DLRQP) for color image processing under the plug-and-play (PnP) framework. Specifically, the quaternion representation with low-rank constraint is introduced to denote the color image in a holistic way and one advanced denoiser is adopted to explore the deep prior in an iterative process. To tightly approximate the quaternion rank, one nonconvex penalty function is further utilized. We derive an alternate iterative approach to tackle the proposed model. We empirically demonstrate that our model can achieve superior performance over existing methods on both color image denoising and inpainting tasks.

Index Terms—Low-rank quaternion representation, deep prior, plug-and-play, color image denoising, color image inpainting.

I. INTRODUCTION

Capturing three color channels and enriching faithful representation of real scenes, color image processing has been widely applied in varied fields, such as color face recognition [1], color image encryption [2], color image denoising [3], [4], color image inpainting [5], [6], [7]. During the acquisition of color images, the image may contain mixed noise or miss some pixels due to the acquisition equipment or various other factors inevitably. In this paper, we mainly focus on two low-level color image processing tasks: color image denoising and color image inpainting, i.e., recovering the latent color image from its noisy observation or incomplete one.

In previous studies, low-rank matrix approximation (LRMA) has been applied in numerous image processing applications [8], [9], [10], [11], [12], [13]. Its general process is to transform the image pixels of one spatial dimension in images as one column vector of the matrix and then utilize the low-rank attribute by an optimization algorithm to restore a clean image. Zhang et al. [10] proposed a modified Schatten-\(p\) norm minimization for accelerated matrix completion problem. Using non-local self-similarity, the low-rank matrix-based image restoration framework [12] was proposed. The multi-matrices low-rank decomposition model was developed for challenging image recovery problem [13]. Based on nonconvex nonsmooth rank (NRR), the work in [14] introduced a novel weighted NNR relaxation. In addition, Wang et al. [15] proposed the group sparse representation based on nonconvex weighted \(\ell_p\) minimization. LRMA generally works well in some grayscale image-based applications [16], while the results based on LRMA methods may plummet when dealing with color images. The reason is that these LRMA methods tackle each color channel separately or concatenate three color channels and would destroy the strong correlation among three color channels [5]. In Fig. 1(a), we show how LRMA splits the three channels of a color image.

As an extension of two-dimensional matrix, tensor structure has gained keen attention in color images and higher-order data applications due to its capacity to preserve information in three dimensions simultaneously [17], [18], [19], [20]. For color images, the pixel values of three RGB color channels can be represented as three frontal slices of a tensor, as shown in Fig. 1(b). That is, one color image would be expressed as a three-order tensor. Most low-rank tensor
approximation (LRTA)-based means can be summarized into two groups: 1) tensor decomposition-based methods; 2) tensor rank minimization-based methods. In the first one, a higher-order tensor can be decomposed into the product of two or more smaller ones, reducing the computational cost in this way [17], [21], [22]. The classic examples are the CANDECOMP/PARAFAC (CP) decomposition based methods [23] and the Tucker decomposition based methods [24]. Recently, Zhao et al. proposed to use tensor train decomposition [25] and denoisers to achieve tensor completion [26]. The second way is generally implemented by the regularization term of the tensor rank approximation, such as the tensor nuclear norm (TNN) [18], [20]. This is because minimizing the tensor rank is an NP-hard problem, and the TNN can be utilized as a compact convex substitution of the tensor rank. Tensor completion methods based on TNN [27], [28] have been extensively studied. Nevertheless, methods of the first type are inevitable to pre-define the tensor rank, but how to determine an accurate rank in practice is still a challenging problem [29]. Methods of the second category overcome the above method, yet with high computational bottleneck.

Another promising way to express color images is the recently popular quaternion representation [30], in which quaternion, as extension of imaginary number, includes one real component and three imaginary components. To seamlessly denote the color images, quaternion representation (QR) uses one pure quaternion number with three imaginary components to store three color channels [31]. In Fig. 1, we show the comparison among low-rank matrix approximation, tensor representation, and quaternion representation. Thus, the main advantage of QR is that the three color channels simultaneously are treated by QR representation, which can well keep the connection between image colors and achieve superior performance in color image processing applications [1], [30], [32], [33], [34], [35], [36], [37]. For example, Xu et al. proposed a novel approach based on quaternion-based dictionary learning model to restore color image [34]. Zou et al. used quaternion collaborative and sparse representation to preserve the color structures [1]. Besides, a new quaternion based sparse regression algorithm was put forward to enhance robustness for two-dimensional quaternion principle component analysis [35]. Compared to tensor representation-based methods, QR-based methods can not only fully utilize color information, but preserve the orthogonal character for the coefficients of three channels, which attains a structured representation [34]. Although current QR-based color image processing methods could represent color images well, they still incorporate some other prior knowledge. The studies in [5], [33], [38], and [39] extended the popular LRMA variants, such as matrix factorization, Geman function, weighted nuclear norm into quaternion representation. Huang et al. [40] proposed the quaternion extension of dictionary learning algorithm and total variation to recovery color image.

Owing to the powerful representation learning capability of deep learning, deep convolutional neural network (CNN)-based grayscale and color image methods have been developed and achieved promising performance [3], [37], [41], [42]. While much of the research into color image denoising and inpainting usually focuses on one specific handcrafted prior such as smoothness [43], low-rankness [32], sparsity [44] or even deep prior [41] but unfortunately fails to fuse deep and handcrafted priors of color images flexibly and simultaneously.

Thus, we consider the possibility to develop one unified method combining two popular color image priors. Meanwhile, the matrix representation for the color image destroys the inherent structural information. Compared to tensor representation, QR is considered to be integrated into this framework since QR preserves the orthogonal character for the coefficients of three channels, which attains a structured representation while preserving structural information. Therefore, we consider introducing quaternion representation to provide a novel way to model color images while using a unified framework of low-rank and deep priors. To achieve this goal, we intend to propose a novel low-rank quaternion representation method with a deep image prior (DLRQP) for color image denoising and inpainting, which integrates low-rank quaternion prior and deep prior into one unified framework. In DLRQP, the low-rank quaternion and deep priors are explored in a mutual promotion manner. Consequently, DLRQP inherits the good interpretability of low-rank prior, powerful structure retention of quaternion representation, and strong learning capability of deep prior. Specifically, DLRQP utilizes the flexible plug-and-play (PnP) framework to incorporate an advanced denoiser and low-rank quaternion representation in an iterative process. Besides, DLRQP also exploits a nonconvex function to obtain a better approximation of the quaternion rank. The contributions of this work mainly include as follows:

- We propose the synchronous integrated deep prior and low-rank quaternion prior method (DLRQP) for color image processing, which probes the global property and local characterization of color images deeply and simultaneously.
- Different from existing QR-based methods that only focus on low-rank property or utilize deep conventional prior, DLRQP integrates a deep learning-based denoiser
and low-rank quaternion representation under the PnP framework. A flexible and efficient solver based on the alternating direction method of multipliers (ADMM) is also developed.

- Concretely speaking, we further extend DLRQP to deal with color image denoising and inpainting tasks. Experiments on color images are carried out and the results clearly confirm that the proposed DLRQP is superior to several competing methods on these two tasks.

The rest parts of this paper are organized as follows. We review several notations and preliminary knowledge in Section II. Section III provides the proposed DLRQP model, extends DLRQP for color image denoising and color image inpainting and devises the ADMM-based algorithms. Section IV conducts experiments to verify the effectiveness of the proposed DLRQP method on color image denoising and inpainting tasks. Finally, this article is concluded in Section V.

II. NOTATIONS AND PRELIMINARIES

In this section, we briefly give several notations and preliminary knowledge about quaternion algebra and plug-and-play framework used in our paper.

A. Notations and Quaternion Algebra

We summarize several necessary notations and real number algebra in Table I. Quaternion space \( \mathbb{H} \), an extension of real and complex spaces, is a mathematical concept invented by Hamilton in 1843 [45]. Given a quaternion number \( a \), we can express it in the form of one real part and three imaginary parts, i.e.,

\[
\hat{a} = a_0 + a_1i + a_2j + a_3k,
\]

where \( a_i \in \mathbb{R} \) \((i = 0, 1, 2, 3)\) are real numbers and \( i, j, k \) denote three imaginary units. In particular, if \( a_0 \) is equal to zero, \( \hat{a} \) is called a pure quaternion. Similar to complex numbers, quaternion addition or subtraction simply add or subtract components, i.e.,

\[
\hat{a} \pm \hat{b} = (a_0 \pm b_0) + (a_1 \pm b_1)i + (a_2 \pm b_2)j + (a_3 \pm b_3)k.
\]

(2)

It should be noted that in general, the product of quaternions does not satisfy the commutative law, i.e., \( \hat{a}\hat{b} \neq \hat{b}\hat{a} \).

Given a quaternion \( \hat{a} = a_0 + a_1i + a_2j + a_3k \), its conjugate and modulus are expressed as \( \hat{a}^* = a_0 - a_1i - a_2j - a_3k \) and

\[
|\hat{a}| = \sqrt{a_0^2 + a_1^2 + a_2^2 + a_3^2},
\]

\( \hat{a}^{-1} \) represents the inverse of \( \hat{a} \), and \( \hat{a}^{-1}\hat{a} = \hat{a}\hat{a}^{-1} = 1 \).

Analogously, given a quaternion matrix \( \hat{A} = A_0 + A_1i + A_2j + A_3k \), where \( A_i \in \mathbb{R}^{M \times N} \) \((i = 0, 1, 2, 3)\), its conjugate, transpose and conjugate transpose are defined as \( \hat{A}^* = (\hat{a}_{i,j})^* \), \( \hat{A}^T = (\hat{a}_{i,j})^T \) and \( \hat{A}^H = (\hat{a}_{i,j}^*)^T \), respectively, where \( 1 \leq i \leq M, 1 \leq j \leq N \). The Frobenius norm of \( \hat{A} \) is denoted as

\[
\|\hat{A}\|_F = \sqrt{\sum_{i=1}^M \sum_{j=1}^N |\hat{a}_{i,j}|^2}.
\]

Quaternion matrices are widely employed to transform them from the quaternion domain into the complex adjoint form, i.e., \( \hat{A} = A_x + A_yj \) [46] where \( A_x, A_y \in \mathbb{C}^{M \times N} \). We use \( \mathcal{P} \) to denote the operation of converting the quaternion form into the complex form, then the complex representation form of \( \hat{A} \in \mathbb{R}^{M \times N} \) is expressed as

\[
\mathcal{P}\hat{A} = \begin{bmatrix} A_x & A_y \n A_y & -A_x^* \end{bmatrix} \in \mathbb{C}^{2M \times 2N}.
\]

(3)

For more details on quaternion algebra, one can refer to [46] and [47]. Following the quaternion singular value decomposition (QSVD) [46], the definitions of quaternion rank and quaternion nuclear norm can be given:

Definition 1 (Quaternion Rank): [5]: For a quaternion matrix \( \hat{A} \in \mathbb{H}^{M \times N} \), the number \( n \) of nonzero singular values is represented as the rank, where \( n < min\{M,N\} \).

Definition 2 (Quaternion Nuclear Norm (QNN)): [5]: QNN of a quaternion matrix \( \hat{A} \in \mathbb{H}^{M \times N} \), expressed as \( \|\hat{A}\|_{QNN} \), is the sum of all nonzero singular values, i.e.,

\[
\|\hat{A}\|_{QNN} = \sum_i \sigma_i(\hat{A}).
\]

Similar to the nuclear norm in the real domain, the quaternion nuclear norm is also the convex substitution of the quaternion rank [5]. To tackle such convex optimization problem on quaternions, we introduce the following theorem:

Theorem 1 (Quaternion Singular Value Thresholding (QSVT)): [5]: Given an arbitrary quaternion matrix \( \hat{A} \in \mathbb{H}^{M \times N} \) and a positive real number \( \lambda \), the QSVT method can be expressed as follows:

\[
\hat{\hat{B}} = \hat{U}\hat{S}_\lambda(\hat{\hat{A}})\hat{V}^H,
\]

(4)

where \( \hat{\hat{U}}, \hat{\hat{V}} \) are from the QSVD of the quaternion matrix \( \hat{\hat{A}} \), and \( \hat{S}_\lambda(\hat{\hat{A}}) \in \mathbb{R}^{M \times N} \) is denoted as \( \hat{S}_\lambda(\hat{\hat{A}}) = \text{diag}(\max(\sigma_i(\hat{\hat{A}}) - \lambda, 0)) \).

B. Low-Rank Matrix Approximation

The main aim of the LRMA is to recover a high-quality image \( \mathbf{X} \) from the noisy observation \( \hat{\mathbf{A}} \), and the optimization problem can be formulated as:

\[
\min_{\mathbf{X}} \lambda \cdot \text{Rank}(\mathbf{X}) + \frac{1}{2} \|\mathbf{X} - \hat{\mathbf{A}}\|_F^2,
\]

(5)

where \( \text{Rank}(\cdot) \) is the rank function, and \( \lambda > 0 \) is a regularization parameter. The first term named regularization term is to impose the low-rank property of the image to
constrain the model. The second term, i.e., data fidelity term constrains the discrepancy between $X$ and $A$ to be as small as possible.

Since the rank minimization problem in Eq. (5) is usually NP-hard, the nuclear norm was put forward to approximate the matrix rank, which proved that the nuclear norm is the tightest convex relaxation [48]. Various heuristic convex and nonconvex surrogates were proposed [11], [49], [50].

C. Plug-and-Play Framework

The PnP framework was first discovered in [51], which utilized advanced denoising algorithms as implicit priors plugged into the forward model, providing a flexible and efficient way to deal with various inverse problems. Under the ADMM framework, all variables could be decoupled into several subproblems. Therefore, the solution process of the ADMM framework has a modular structure, in which the subproblem utilized to deal with the prior regularization term can be written as:

$$
x = \arg \min_x \Phi(x) + \frac{1}{2\sigma^2} \|x - y\|^2,
$$

where $\Phi(x)$ represents the prior term; $y$ can be regarded as a noisy image. Eq. (6) can be expressed as a denoising problem. Therefore, this denoising module can be replaced with the off-the-shelf denoising algorithm. The PnP framework has shown excellent reconstruction capability in numerous applications, including deblurring [52], restoration [53], [54], super-resolution [55] and demosaicing [56].

III. THE PROPOSED DEEP AND LOW-RANK QUATERNION PRIORS (DLRQP) MODEL

We first propose the DLRQP model based on deep and nonconvex low-rank quaternion priors. Then, we extend the proposed model into two classic color image processing tasks, namely color image denoising and color image inpainting, and propose the ADMM-based algorithms.

A. The Proposed DLRQP Model

When handling color image processing tasks, previous LRMA method in Eq. (5) usually exploited either monochromatic model or the concatenation model. That is, the former model processes each color channel of color images independently while the second model concatenates three color channels as one unified one and finally they used existing grayscale image processing methods directly without specific operations. However, they would destroy the inherent correlation among three color channels since they cannot represent color images holistically. Although the improvement has been made by the recently proposed low-rank quaternion approximation [11] and CNN-based methods [41], these methods focus on one specific prior knowledge such as smoothness [43], low-rankness [32], or even deep prior [41], but unfortunately fail to fuse deep and handcrafted priors of color images flexibly.

Meanwhile, we consider utilizing the prior knowledge, which is obtained from deep learning-based ways and low-rank quaternion representation. Inspired by the huge success of Convolutional Neural Networks (CNNs) in image restoration, we consider incorporating the powerful strength of CNNs to capture details as a compensation for restoring underlying images. In this paper, we intend to integrate the strengths between low-rank priors based on quaternion representation and deep learning-based image priors. As a consequence, our DLRQP model can be expressed as follows:

$$
\min_{\tilde{X}} \lambda \cdot \text{Rank}(\tilde{X}) + \alpha \Phi(\lambda) + \frac{1}{2} \|\tilde{X} - \hat{A}\|_F^2, \tag{7}
$$

where $\alpha > 0$ is a regularization parameter, and $\lambda \in \mathbb{R}^{M \times N}$ represents the tensor form of the recovered color image. $\tilde{X} \in \mathbb{H}^{M \times N}$ and $\hat{A} \in \mathbb{H}^{M \times N}$ represent the constructed quaternion matrix and the observed quaternion matrix, respectively. The tensor $\lambda$ and the quaternion matrix $\tilde{X}$ are different representations of the same color image. We can define an operator $h : \mathbb{R}^{M \times N} \rightarrow \mathbb{H}^{M \times N}$, which enables the conversion between tensor and quaternion, i.e., $\tilde{X} = h(\lambda)$ and $\lambda = h^{-1}(\tilde{X})$. $h^{-1}$ is the inverse of $h$. Specifically, this operator can be expressed as $\tilde{X} = \lambda (\cdot, \cdot, 1) \mathbf{i} + \lambda (\cdot, \cdot, 2) \mathbf{j} + \lambda (\cdot, \cdot, 3) \mathbf{k}$.

RANK() denotes the quaternion rank function. $\Phi(\lambda)$ denotes an implicit regularizer, and we can plug an advanced denoiser as deep image prior. Instead of using CNNs to build network structures and perform end-to-end training, a denoising CNN trained on a large number of natural images as the implicit prior is plugged in existing model directly. In summary, our model is more flexible and economical to deal with different application scenarios.

To tackle the NP-hard issue, we adopt the Laplace function as the nonconvex rank function to better approximate the quaternion rank and further formulate Eq. (7) as

$$
\min_{\tilde{X}} \|	ilde{X}\|_{\gamma,QNN} + \alpha \Phi(\lambda) + \frac{1}{2} \|	ilde{X} - \hat{A}\|_F^2, \tag{8}
$$

where $\gamma > 0$ is a constant, which is the Laplace parameter. Specifically, the Laplace function in QNN can be expressed as follows:

$$
\|\tilde{X}\|_{\gamma,QNN} = \sum_i \phi(\sigma_i(\tilde{X}), \gamma) \quad = \sum_i \phi(1 - e^{-\sigma_i(\tilde{X})/\gamma}), \tag{9}
$$

where $\phi(x) = 1 - e^{-x/\gamma}$ denotes the Laplace function, and $\sigma_i(\cdot)$ denotes the $i$-th singular value of the quaternion rank. Next, we discuss the concrete representation of the proposed DLRQP model in specific color image processing tasks.

B. DLRQP for Color Image Denoising

We first use the proposed DLRQP model to solve the color image denoising task, whose main aim is to estimate the original color image $\lambda$ by restraining the noise $\epsilon$ from the observed color image $A$. The corresponding model is expressed as:

$$
\hat{A} = \lambda + \epsilon. \tag{10}
$$

In generally, $\epsilon$ is the additive white Gaussian noise, and its mean is zero and the variance is $\sigma^2$. Since this type
of inversion problem is ill-posed, some priors should be explored as regularization terms in the solution model to obtain satisfactory denoising performance. In this part, we propose a novel denoising model, which simultaneously considers local, nonlocal, and global prior information of color images. In the proposed model, we employ nonlocal self-similarity (NSS) to characterize the nonlocal information of color images. NSS is widely utilized in image processing problems by clustering similar patches into a group [57], [58]. Zha et al. further proposed to combine NSS and group sparsity residual constraint to deal with various image processing tasks [59]. Xie et al. utilized pixel weighting strategy and NSS to explore the global and nonlocal information of the image [60]. In this subsection, we consider quaternion representation combined with NSS to construct nonlocal priors for images. Specifically, the original color image is divided into many small patches with size of $n_1 \times n_2 \times 3$. By using the operator $h$, we represent each patch as a quaternion matrix and stretch it into a column vector, i.e., $\mathbf{a}_i \in \mathbb{H}^{n_1 n_2}$, and its similar patch group is constructed by searching for the $m$ closest image patches within a given searching window. It is worth noting that the $m$ similar patches contain a similar patch group. NSS follows the assumption that there exist many image patches similar to the exemplar patch in the whole image. Moreover, we introduce the deep prior is investigated on the deep learning-based denoiser under the PnP framework, which can benefit from each other. Meanwhile, we utilize an implicit regularization term on the similar patch group $\mathbf{a}_i$. A quaternion matrix, $\mathbf{A}_i = [\mathbf{a}_{i1}, \mathbf{a}_{i2}, \ldots, \mathbf{a}_{im}] \in \mathbb{H}^{n_1 n_2 \times m}$, is utilized to store these similar patches.

Global correlation is characterized by the low-rankness of similar patch groups. NSS follows the assumption that there exist many image patches similar to the exemplar patch in the natural image. Based on this, we also assume that the quaternion matrix $\mathbf{A}_i$ we constructed as aforementioned is of low rank. Meanwhile, we utilize an implicit regularization term on the whole image to plug a denoising CNN to preserve local details. By formulating a novel color image denoising framework in the quaternion domain that can use these local, nonlocal, and global priors simultaneously, we give the following optimization

$$
\min_{\hat{\mathbf{X}}} \|\hat{\mathbf{X}}\|_{\gamma,\text{QNN}} + \alpha \Phi(\mathbf{X}) + \frac{1}{\tau^2} \|\hat{\mathbf{X}} - \mathbf{A}\|_F^2,
$$

(11)

where $\|\hat{\mathbf{X}} - \mathbf{A}\|_F^2$ is the data fidelity term utilized to constrain the model to conform to the image degradation process in Eq. (10). Different from Eq. (7), here $\mathbf{X} \in \mathbb{H}^{n_1 n_2 \times m}$ represents a quaternion matrix composed of similar patches. In addition, the $L_2$-norm has a good performance in suppressing Gaussian noise. $\Phi(\mathbf{X})$ is an implicit regularization term, which is tackled by the denoising CNN under the PnP framework. Different from existing denoising methods, our proposed model is based on quaternion representation, which can better characterize the structural information of color images. Then we explore three different types of priors that simultaneously leverage the model so that they can benefit from each other. Moreover, we introduce the deep learning-based denoiser under the PnP framework, which has greater flexibility and generalization ability applied to different scenarios and applications than directly designing a network structure. The proposed DLQRP model in (11) imposes the low-rank constraint on the similar patch group $\hat{\mathbf{X}}$ to explore the NSS property while the deep prior is investigated on the whole image $\mathbf{X}$.

Then, we utilize ADMM and PnP framework to solve the problem (11). An auxiliary variable $\mathbf{Z} \in \mathbb{R}^{M \times N \times 3}$ is introduced, and the minimization problem (11) can be rewritten as

$$
\min_{\mathbf{X}, \mathbf{Z}, \mathbf{A}} \|\hat{\mathbf{X}}\|_{\gamma,\text{QNN}} + \alpha \Phi(\mathbf{X}) + \frac{1}{\tau^2} \|\mathbf{X} - \hat{\mathbf{A}}\|_F^2,
$$

s.t. $\quad \mathbf{Z} = \mathbf{X}.
$$

(12)

The relevant augmented Lagrangian function of Eq. (12) is expressed as

$$
\mathcal{L}_\mu(\mathbf{X}, \mathbf{Z}; \Lambda) = \|\hat{\mathbf{X}}\|_{\gamma,\text{QNN}} + \alpha \Phi(\mathbf{X}) + \frac{1}{\tau^2} \|\mathbf{X} - \hat{\mathbf{A}}\|_F^2 + \frac{\mu}{2} \|\mathbf{Z} - \mathbf{X} + \frac{\Lambda}{\mu}\|_F^2,
$$

(13)

where $\Lambda \in \mathbb{R}^{M \times N \times 3}$ and $\mu$ are the Lagrangian multiplier and the penalty parameter, respectively. Then, Eq. (13) can be tackled by updating all variables iteratively.

**X sub-problem:** The $\hat{\mathbf{X}}$ at $k$-th iteration is updated by

$$
\hat{\mathbf{X}}^{k+1} = \arg \min_{\mathbf{X}} \|\hat{\mathbf{X}}\|_{\gamma,\text{QNN}} + \frac{1}{\tau^2} \|\hat{\mathbf{X}} - \hat{\mathbf{A}}\|_F^2 + \frac{\mu}{2} \|\mathbf{Z} - \mathbf{X} + \frac{\Lambda}{\mu}\|_F^2,
$$

(14)

where $\hat{\mathbf{X}} \in \mathbb{H}^{n_1 n_2 \times m}$ represents a quaternion matrix composed of nonlocal similar patches of the exemplar patch, and $\mathbf{X} \in \mathbb{R}^{M \times N \times 3}$ is the full-sized color image. For ease of calculation, when updating $\mathbf{X}$, we also take the nonlocal similar patches with the same position and number for $\mathbf{X}, \mathbf{Z}, \Lambda$, to construct a quaternion matrix, then Eq. (14) can be rewritten as

$$
\hat{\mathbf{X}}^{k+1} = \arg \min_{\mathbf{X}} \|\hat{\mathbf{X}}\|_{\gamma,\text{QNN}} + \frac{1}{\tau^2} \|\hat{\mathbf{X}} - \hat{\mathbf{A}}\|_F^2 + \frac{\mu}{2} \|\mathbf{Z} - \mathbf{X} + \frac{\Lambda}{\mu}\|_F^2 + \beta \|\mathbf{X} - \mu^k \mathbf{Z} + \frac{\Lambda^k}{\mu^k} - \frac{\mu^k \mathbf{Z} + \lambda^k}{\mu^k}\|_F^2,
$$

(15)

where $\beta = \frac{2 + \tau^2}{2 + \tau^2}$.

Then, we introduce the following Theorem to give the solution to the above optimization.

**Theorem 2:** [5]: Given a quaternion matrix $\mathbf{B} \in \mathbb{H}^{M \times N}$, its QSVD is defined as $\mathbf{B} = \mathbf{USV}^H$ and $\tilde{\mathbf{S}} = \text{diag}(\sigma_{\mathbf{B}})$. The global optimal solution of Eq. (16) is $\hat{\mathbf{X}} = \mathbf{USV}^H$, where $\tilde{\mathbf{S}} = \text{diag}(\tilde{\sigma})$. The $\tilde{\sigma}$ can be acquired by

$$
\tilde{\sigma} = \arg \min_{\sigma \geq 0} \lambda \|\sigma\|_2 + \frac{1}{2} \|\sigma - \sigma_{\mathbf{B}}\|_2^2.
$$

(17)

Since $\phi(\cdot)$ is the nonconvex function, which is continuous, differentiable, monotonically nondecreasing on $[0, +\infty)$, and

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concave with respect to \( \sigma \), it is not easy to tackle the problem (17) directly. The DCA aims to iteratively optimize the difference of two convex functions by linearizing the concave term, which is transformed from the original non-convex function. Heuristically, we utilize the DCA to tackle Eq. (17) and in the \((k+1)\)-th iteration we obtain the following formula:

\[
\sigma^{k+1} = \arg \min_{\sigma \geq 0} \lambda \ast \nabla \phi(\sigma{k})^T \sigma + \frac{1}{2} \| \sigma - \sigma_k \|_2^2,
\]

(18)

where \( \nabla \phi(\sigma{k})^T \) denotes the gradient of \( \phi(\cdot) \) at \( \sigma{k} \). We can get the following closed-form solution of Eq. (18)

\[
\sigma^{k+1} = (\sigma_k - \lambda \ast \nabla \phi(\sigma^k))_+.
\]

(19)

After repetitious iterations, DCA will eventually converge to \( \hat{\sigma} \) which is the local optimal point, and then \( \hat{X} = \hat{U} \text{diag}(\hat{\sigma}) \hat{V}^H \).

The entire procedure of handling the Eq. (16) is summarized in Algorithm 1.

Algorithm 1 Solving Problem (16) Using DCA

Input: The quaternion matrix \( \hat{B} \in \mathbb{H}^{M \times N} \), parameter \( \lambda > 0 \).

Initialization: \( k = 0, \sigma = 0 \).

1: repeat
2: Calculate the gradient of \( \phi(\cdot) \);
3: Update \( \sigma^{k+1} \) via Eq. (19);
4: until converge

Output: The reconstructed tensor \( \hat{X} \).

\( Z \) sub-problem: After Eq. (15) is updated, we get a clean patch group \( \hat{X} \) and then convert \( \hat{X}, \hat{Z}, \hat{A} \) into real number matrices, and concatenate all the patches to obtain \( \hat{X}, \hat{Z}, \hat{A} \) with the same size of the original image, respectively. Fixing the rest variables, \( \hat{Z} \) is updated by addressing the subproblem

\[
\hat{Z}^{k+1} = \arg \min_{\hat{Z}} \alpha \Phi(\hat{Z}) + \frac{\mu}{2} \| \hat{Z} - (\hat{X}^{k+1} + \hat{A}^{k})/\mu \|_F^2.
\]

(20)

Then, a flexible denoising CNN, i.e., FFDNet [41], is utilized as a deep denoiser. Taking \( \hat{X}^{k+1} + \hat{A}^{k}/\mu \) as the input of FFDNet, the solution of Eq. (20) can be obtained as

\[
\hat{Z}^{k+1} = \text{FFDNet}(\hat{X}^{k+1} + \hat{A}^{k}/\mu, \beta),
\]

(21)

where \( \beta = \sqrt{\alpha/\mu} \). Under the PnP framework, \( \hat{X}^{k+1} + \hat{A}^{k}/\mu \) is taken as the noisy image, and the denoising result \( \hat{Z} \) is obtained by the plugged prior.

Update \( \hat{A} \) and \( \hat{\mu} \): For the multiplier \( \hat{A} \), it is updated by

\[
\hat{A}^{k+1} = \hat{A}^{k} + \hat{\mu}^{k}(\hat{X}^{k+1} - \hat{Z}^{k+1}),
\]

(22)

And the parameter \( \hat{\mu} \) can be updated by

\[
\hat{\mu}^{k+1} = \min\{\rho \ast \hat{\mu}^{k}, \text{Max}\},
\]

(23)

where \( \rho \) denotes an adjustable parameters, and \( \text{Max} \) denotes the maximum value of the \( \mu \). Inspired by [49] and [32], the iterative regularization scheme is introduced, which is expressed as

\[
\hat{A}^{k+1} = \hat{A}^{k} + \theta(\hat{A} - \hat{A}^{k}),
\]

(24)

where \( k \) is the iteration number, and \( \theta \) denotes the relaxation parameter. Finally, the whole algorithm procedure is summarized in Algorithm 2.

Algorithm 2 DLRQP for Color Image Denoising

Input: The noisy image \( \mathcal{A} \in \mathbb{R}^{M \times N \times 3} \), the parameter \( \mu, \alpha \), relaxation parameter \( \theta \), noise level \( r \).

Initialization: \( \mathcal{X}^0 = \mathcal{Z}^0 = \mathcal{A}, \mathcal{A}^0 = \mathcal{Z}^0 = \mathcal{A} \), \( \mu_0 = 0.001, \rho = 1.1 \).

1: for \( k = 0 : K \) do
2: Iterative regularization via Eq. (24);
3: for \( a \in \mathcal{A}^k \) do
4: Build similar patch group \( \mathcal{A}_i, \mathcal{Z}_i, \mathcal{A}_i \) and convert them to quaternion matrices \( \mathcal{A}_i, \mathcal{Z}_i, \mathcal{A}_i \);
5: Update \( \hat{X}_i \) via Eq. (15);
6: Convert \( \hat{X}_i, \hat{Z}_i, \hat{R}_i \) into real matrices \( \hat{X}_i, \hat{Z}_i, \hat{A}_i \);
7: end for
8: Aggregate \( \{\mathcal{X}_i\}, \{\mathcal{Z}_i\}, \{\mathcal{A}_i\} \) to construct the complete images \( \mathcal{X}^k, \mathcal{Z}^k \), and the multiplier \( \mathcal{A}^k \);
9: Update \( \mathcal{Z}^{k+1} \) via Eq. (22);
10: Update \( \mathcal{A}^{k+1} \) via Eq. (23);
11: end for

Output: The denoised image \( \mathcal{X}^k \).

C. DLRQP for Color Image Inpainting

As aforementioned, another classic image processing task is color image inpainting, whose aim is to reconstruct a complete image from observation with missing pixels. Adopting the proposed low-rank quaternion representation incorporated with deep image prior, our color image inpainting model is written as follows

\[
\min_{\mathcal{X}} \| \mathcal{X} \|_{\gamma, \text{QNN}} + \lambda \Phi(\mathcal{X}),
\]

(25)

s.t. \( \mathcal{P}_\Omega(\mathcal{X}) = \mathcal{P}_\Omega(\mathcal{O}) \),

where \( \mathcal{X} \in \mathbb{H}^{M \times N} \) indicates the reconstructed image, and \( \hat{\Omega} \) denotes the incomplete observation. \( \Omega \) denotes the index set of the observed elements. The operation \( \mathcal{P}_\Omega(\cdot) \) denotes the projection function, which retains the elements in \( \Omega \) consistent, while the rest be zeros. \( \mathcal{X} \) is the tensor representation of the color image, and \( \Phi(\mathcal{X}) \) is an implicit regularizer to plug a deep denoiser. This model combines the superior global information construction capability of quaternion representation with the advantages of advanced deep denoisers for capturing local details. Due to missing pixels, the NSS property cannot be guaranteed. Thus, the low-rank restriction is imposed on the whole color image.

To facilitate computation, we introduce the auxiliary variable \( \mathcal{Z} \in \mathbb{R}^{M \times N \times 3} \) and one auxiliary quaternion matrix variable \( \mathcal{E} \in \mathbb{H}^{M \times N} \), and the Eq. (25) is reformulated as

\[
\min_{\mathcal{X}, \mathcal{E}, \mathcal{Z}} \| \mathcal{X} \|_{\gamma, \text{QNN}} + \lambda \Phi(\mathcal{Z}),
\]

s.t. \( \mathcal{Z} = \mathcal{X}, \mathcal{X} + \mathcal{E} = \hat{\Omega}, \mathcal{P}_\Omega(\mathcal{E}) = 0 \).
The corresponding augmented Lagrangian function of the problem (26) is
\[
\mathcal{L}(\hat{X}, \hat{E}, Z; M, Y) = \|\hat{X}\|_{\gamma, QNN} + \lambda \Phi(Z) \\
+ \frac{\mu}{2} \|\hat{O} - \hat{X} - \hat{E} + \frac{M^k}{\mu} + \|X - Z + \frac{Y}{\mu}\|^2_F, \\
\text{(27)}
\]
where \(M\) and \(Y\) indicates the Lagrangian multipliers. \(\mu\) denotes a parameter. Then, an ADMM-based solution procedure is developed to solve the minimization problem under the PnP framework. We solve the above problem by update each variable alternately as follows:
\[
\hat{X}^{k+1} = \arg\min_X \|\hat{X}\|_{\gamma, QNN} + \frac{\mu}{2} \|\hat{O} - \hat{X} - \hat{E} + \frac{M^k}{\mu}\|^2_F \\
+ \frac{\mu}{2} \|\hat{X} - Z^k + \frac{Y^k}{\mu}\|^2_F, \\
\text{(28)}
\]
\[
\hat{E}^{k+1} = \arg\min_{\hat{E}} \|\hat{O} - \hat{X}^{k+1} - \hat{E} + \frac{M^k}{\mu}\|^2_F, \\
\text{subject to } \mathcal{P}_\Omega(\hat{E}) = 0. \\
\text{(29)}
\]
\[
Z^{k+1} = \arg\min_Z \lambda \Phi(Z) + \frac{\mu}{2} \|X^k - Z + \frac{Y}{\mu}\|^2_F, \\
\text{(30)}
\]
\[
\hat{M}^{k+1} = \hat{M}^k + \mu \cdot \frac{\hat{O} - \hat{X}^{k+1} - \hat{E}^{k+1}}{\mu}, \\
\text{(31)}
\]
\[
\hat{Y}^{k+1} = \hat{Y}^k + \mu \cdot \frac{\hat{X}^{k+1} - Z^{k+1}}{\mu}, \\
\text{(32)}
\]
\[
\mu^{k+1} = \min(\mu \cdot \mu^k, \text{Max}). \\
\text{(33)}
\]
Specifically, for the \(\hat{X}\) subproblem in (28), the preprocessing step here is to convert the tensors \(X\), \(Z\), and \(A\) into quaternion representations using the \(h\) operator. Then, the \(\hat{X}\) subproblem can be rewritten as
\[
\hat{X}^{k+1} = \arg\min_X \|\hat{X}\|_{\gamma, QNN} \\
+ \mu \|\hat{X} - \frac{\hat{O} - \hat{E}^k + \hat{M}^k}{\mu} + \frac{\hat{Z}^k - \hat{Y}^k}{\mu}\|^2_F, \\
\text{(34)}
\]
The closed-form solution of Eq. (34) is obtained by Algorithm 1. The subproblem in (29) can be tackled by \(\hat{O} - \hat{X}^{k+1} + \hat{M}^k/\mu\). Meanwhile, we utilize the constraint \(\mathcal{P}_\Omega(\hat{E}) = 0\) throughout the iteration to ensure that the observed pixels will not be changed. The \(Z\) subproblem in (30) can be solved by
\[
Z^{k+1} = \text{FFDNet}(\hat{X}^{k+1} + \frac{\hat{Y}^{k+1}}{\mu} \cdot \sqrt{\lambda/\mu^k}). \\
\text{(35)}
\]
The details about the proposed DLRQP for color image inpainting are outlined in Algorithm 3.

D. Differences From Existing Methods

In this subsection, we aim to discuss the differences among our proposed approach and five closely related methods abbreviated as PNNM [12], SSLRDM [13], DSRDP [44], GLON [26] and DP3LRTC [27].

A common shortcoming of the PNNM and SSLRDM methods is that both of them use matrix to represent image data. The color image needs to be concatenated or sliced into matrix, and then input into the models. This way of processing three color channels separately will cause structure information loss. The CNN-based DSRDP method is inevitable to use a large number of images to train the model first. The tensor representation is used to reconstruct the tensor data in GLON and DP3LRTC. However, these approaches are not general enough to deal with color image denoising task.

Instead of tensor representation, our proposed DLRQP method integrates low-rank quaternion representation and deep prior into one unified framework without training for color image denoising and inpainting. Compared with matrix representation, the quaternion representation can process three color channels of the color image at the same time, which realizes the preservation of the inherent structural information of the color image. Compared with tensor representation, QR preserves the orthogonal character for the coefficients of three channels, which attains a structured representation [34]. Meanwhile, the low-rank quaternion and deep denoiser in the unified framework provide global and local priors to complement each other. Compared with only using traditional priors or deep priors, our proposed method can deal with complex scenarios more flexibly without pre-training. In short, the proposed DLRQP significantly differs from the above related methods.

E. Computational Complexity

In this section, we discuss the computational complexity of the proposed DLRQP model. For color image denoising, the time for computing subproblem \(\hat{X}\) is mainly consumed by quaternion SVD, and the complexity is \(O(p \min(n_1 n_2^2, n_1^2 n_2))\), where \(p\) denotes the number of patches; the complexity of computing subproblem \(Z\) is \(O(n_1 n_2 n_1 n_k)\), where \(n_1\) and \(n_2\) are the number of layers, features, and kernel pixels; the computational complexity of the multiplier \(\lambda\) is \(O(n_1 n_2)\). Thus, the total cost of DLRQP for color image denoising is \(O(p \min(n_1^2 n_2^2, n_1 n_2 n_1 n_k + n_1 n_2))\). Analogously, the
TABLE II
DENOISING ASSESSMENT (PSNR/SSIM) OF THE PROPOSED AND COMPETING METHODS

<table>
<thead>
<tr>
<th>Methods</th>
<th>LSCD</th>
<th>KQSVD</th>
<th>WNNM</th>
<th>LRQA</th>
<th>PFDNet</th>
<th>TLRSR</th>
<th>GSHOSVD</th>
<th>Our</th>
</tr>
</thead>
<tbody>
<tr>
<td>Img1</td>
<td>34.470/934</td>
<td>33.810/927</td>
<td>34.660/938</td>
<td>34.770/941</td>
<td>33.590/935</td>
<td>30.980/938</td>
<td>30.680/937</td>
<td><strong>38.870/955</strong></td>
</tr>
<tr>
<td>Img3</td>
<td>32.730/906</td>
<td>32.780/913</td>
<td>34.780/945</td>
<td>34.990/948</td>
<td>36.040/955</td>
<td>33.550/953</td>
<td>31.730/878</td>
<td><strong>36.190/958</strong></td>
</tr>
<tr>
<td>Img4</td>
<td>31.080/922</td>
<td>30.900/918</td>
<td>32.950/940</td>
<td>33.110/944</td>
<td>34.110/957</td>
<td>34.040/977</td>
<td>29.690/912</td>
<td><strong>34.200/958</strong></td>
</tr>
<tr>
<td>Img5</td>
<td>33.970/862</td>
<td>34.200/864</td>
<td>35.780/989</td>
<td><strong>36.030/907</strong></td>
<td>34.660/899</td>
<td>32.480/920</td>
<td>31.670/839</td>
<td><strong>35.010/906</strong></td>
</tr>
<tr>
<td>Img6</td>
<td>34.730/928</td>
<td>34.200/922</td>
<td>35.230/934</td>
<td>35.410/939</td>
<td>34.720/949</td>
<td>32.060/939</td>
<td>31.220/876</td>
<td><strong>36.770/950</strong></td>
</tr>
<tr>
<td>Img7</td>
<td>33.160/944</td>
<td>33.120/942</td>
<td>34.670/958</td>
<td>34.740/959</td>
<td>36.180/967</td>
<td>31.040/950</td>
<td>30.800/913</td>
<td><strong>36.310/968</strong></td>
</tr>
<tr>
<td>Img9</td>
<td>33.470/877</td>
<td>33.610/884</td>
<td>34.050/888</td>
<td>34.160/891</td>
<td>32.900/913</td>
<td>31.600/899</td>
<td>30.400/860</td>
<td><strong>35.380/918</strong></td>
</tr>
<tr>
<td>Img11</td>
<td>34.240/940</td>
<td>33.340/931</td>
<td>34.370/943</td>
<td>34.570/947</td>
<td>36.740/966</td>
<td>32.390/953</td>
<td>30.910/894</td>
<td><strong>36.760/966</strong></td>
</tr>
<tr>
<td>Img12</td>
<td>32.600/929</td>
<td>31.110/917</td>
<td>33.930/940</td>
<td>34.020/943</td>
<td>35.020/955</td>
<td>26.790/901</td>
<td>28.860/883</td>
<td><strong>30.080/957</strong></td>
</tr>
<tr>
<td>Aver.</td>
<td>32.915</td>
<td>32.498</td>
<td>34.055</td>
<td>34.953</td>
<td>35.053</td>
<td>30.335</td>
<td>30.298</td>
<td><strong>35.180</strong></td>
</tr>
</tbody>
</table>

A. Quantitative Assessment

The performance of the DLRQP is quantitatively evaluated by two evaluation indicators, i.e., peak signal-to-noise ratio (PSNR), and structural similarity index (SSIM) [62].

In general, the larger the values of PSNR and SSIM, the better the performance of the algorithm.

B. Artificial Noisy Color Image Denoising

1) Color Image Data: The proposed algorithm is first evaluated on 12 natural RGB color images with the resolution of 256 × 256 as shown in Fig. 2, all of which are added with the additive Gaussian noise with mean zero and variance $\sigma^2$ as simulated noise. Here we test the case where the noise levels $\sigma$ are equal to 10, 30, and 50, respectively.

2) Compared Methods: We compare our proposed DLRQP with the following seven denoising methods, including LSCD [63], KQSVD [34], WNNM [64], LRQA [5], FFDNet [41], TLRSR [65], GSHOSVD [66]. Specifically, KQSVD and LRQA are both quaternion-based denoising methods, where LRQA represents weighted Schatten norm-based LRQA methods with the best denoising perfor-
mance in [5]. LSCD and WNNM are LRMA-based denoising methods. FFDNet is the deep learning-based method. TLRSR and GSHOSVD are tensor-based denoising methods.

3) Parameter Settings: For all comparison methods, we use the recommended parameter settings. In our proposed Algorithm 2, parameters that need to be considered include the patch size $p$, the number of similar patch groups $m$, the number of iterations $K$, and the iterative relaxation coefficient $\theta$. In the case of $\tau = 10, 30, 50$, $p$ is set to $6 \times 6$, $7 \times 7$, $8 \times 8$, and we set $m$ to 70, 90, and 120, respectively. $K$ and $\theta$ are fixed at 4 and 0.1, respectively.

Table II summarizes the PSNR and SSIM values of different denoising approaches under different noise levels. We use bold font to emphasize the best results and underline the second-best ones. Moreover, we calculate the average PSNR values for all methods at different noise levels. Some of experimental results in the Table II are quoted from [5]. It can be observed that our proposed DLRQP acquires the optimal performance in most cases. DLRQP has achieved 0.127dB, 0.149dB, and 0.112dB improvement over the runner-up FFDNet with respect to $\tau = 10, 30, 50$, respectively. This directly demonstrates the advantage of the proposed DLRQP for color image denoising. It can be found that FFDNet outperforms most traditional denoising methods. The reason is that FFDNet has powerful representation learning ability due to its complex network structure. The tensor-based methods TLRSR and GSHOSVD cannot outperform the quaternion-based method, we guess because tensor-based methods do not exploit nonlocal information. Besides, compared with the other two quaternion-based denoising methods, our method improves the average PSNR value about 2dB over KQSVD and about 1dB over LRQA. The above observation confirms the effectiveness of our proposed DLRQP.

Moreover, the visual results of color image denoising for all methods at noise level $\tau = 50$ are shown in Fig. 3. From the enlarged regions, we can observe that there is still obvious unexpected noise in the recovered images by LSCD, TLRSR and GSHOSVD. WNNM and LRQA denoise better, but still lose some details and textures. Compared with FFDNet, our method recovers more details in some regions with low pixel values and obtains higher PSNR values. In terms of visual effects, the proposed DLRQP outperforms other competing methods.

C. Real Noisy Color Image Denoising
We evaluate the proposed DLRQP on real images for the denoising task. We selected two images from real-world noisy dataset RN15 [67], namely Frog and Pattern. We compare the proposed DLRQP method with MCWNNM [32], FFDNet [41], LRQA [5]. We need to estimate the noise level on the real image first. Here we adopt the noise estimation method in [68] for each color channel. According to the best settings given in the FFDNet, we set the input noise to 15 for Frog and 40 for Pattern, respectively. For MCWNNM and LRQA, we set the patch size $p$ is set to 6, the number of nonlocal similar patches $m$ is set to 70, and the searching window is set to $20 \times 20$.

Since the ground truth cannot be obtained for real noisy images experiments, we only report the comparison of visual restoration results in Fig. 4. It can be observed that all methods recover clean images. However, the images denoised by MCWNNM and LRQA still exist some blur and noise, while the denoising results of our proposed method are clearer than that of other methods. Meanwhile, our method has fewer staircase artifacts on details than FFDNet.

D. Color Image Inpainting
In this section, we conduct color image inpainting experiments to evaluate the performance of the proposed DLRQP.
TABLE III

INPAINTING ASSESSMENT (PSNR/SSIM) OF THE PROPOSED AND COMPETING METHODS

<table>
<thead>
<tr>
<th>Images:</th>
<th>Methods</th>
<th>LMaFit</th>
<th>TNRR</th>
<th>WNNM</th>
<th>MC-NC</th>
<th>D-N</th>
<th>LRQA</th>
<th>LRQMC</th>
<th>PnP-DIP</th>
<th>Our</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aver.</td>
<td>21.310</td>
<td>30.310</td>
<td>27.200</td>
<td>27.870</td>
<td>21.840</td>
<td>30.583</td>
<td>29.481</td>
<td>29.638</td>
<td>33.005</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 5. The 12 RGB color images used in our inpainting experiment. The images are named Img1 to Img12 from left to right.

The selected 12 color images are shown in Fig. 5. The image inpainting methods chosen for comparison include LRMA-based LMaFit [69], TNRR [8], WNNM [49], MC-NC [70], D-N [71], quaternion-based LRQA [5], LRQMC [33], and plug-and-play-based PnP-DIP [72]. Among them, LRQA in [5] represents the LRQA model using the Laplace function, which achieves relatively optimal performance in the color image inpainting task.

Quantitative evaluations of inpainting performance from all methods are presented in Table III. We test three missing rate (MR) cases, namely MR = 0.5, 0.75, 0.85. It is observed that the proposed DLRQP achieves the best performance in most cases. Compared with the quaternion-based inpainting methods LRQA and LRQMC, DLRQP improves the average PSNR at least 2.5dB since DLQRQ also takes the deep prior of color images into consideration. In addition, we find that the performance of the LRMA-based method MC-NC drops sharply as the missing rate increases. In contrast, the proposed DLRQP has better stability. Driven by the powerful structure retention of quaternion representation, DLRQP has surpassed the PnP-based color image inpainting method PnP-DIP, by a large margin. The corresponding improvement is at least 2.6dB. It is worth noting that the improvement of DLRQP on color image inpainting is much obvious than that on color image denoising. We attribute this to the use of nonlocal self-similarity, which may bring the loss of some structural information of color images.

Fig. 6 shows the comparison of visual effects of all methods with MR = 0.85. As we can see, LMaFit and D-N have poor recovery performance at high missing rates. TNRR, WNNM, MC-NC, and LRQMC recover rough structures, but the images still exist obvious missing pixels. LRQA and PnP-DIP
recover better results, but also have some loss of details. The results of PnP-DIP are over-smooth. In comparison, the proposed DLRQP achieves the most visually desirable results with pleasing image details and neat textures. This also confirms the effectiveness of the proposed model combining nonconvex low-rank approximation and FFDNet denoiser in the quaternion domain.

E. Discussion

1) Parameter Selection: We test the influence of the parameters $\gamma$ and $\alpha$ in our proposed DLRQP. In the proposed model, $\gamma$ and $\alpha$ control the weights of the low-rank prior and deep prior, respectively. If $\gamma$ is not suitable for the proposed model, it will affect the integrality of the restored image. If $\alpha$ is too large or too small, it will compromise the local smoothness and details of the image. Based on this, we conduct related experiments on color image denoising with $\tau = 50$. We set $\gamma$ to [0.1, 1] with an increment of 0.1, and set $\alpha$ from $2E+3$ to $3E+3$ with an increment of $1E+2$. Fig. 7 shows the effect of different $\gamma$ and $\alpha$ on PSNR and SSIM for color image denoising. It can be found that the adjustment of parameter $\alpha$ has a greater impact on the results. Satisfactory experimental results can be obtained when $\alpha = [2.4E+3, 2.6E+3]$ for color image denoising. Our DLRQP is insensitive to parameter $\gamma$, which shows the robustness of the nonconvex low-rank prior.

2) Choice of Denoiser for DLRQP: To investigate the effect of different denoisers for our method, we compare the denoising performance of BM3D [73], CBM3D [74], DIP [75], and FFDNet [41] on two images with noise level $\tau = 30$. Visual and numerical results are shown in Fig. 8. We can see that the images recovered by BM3D are over smooth, and CBM3D and DIP methods suffer from blurring in details. Note that the SSIM values of DIP is higher than that of FFDNet. We attribute this to that the particular generator network structure in DIP is beneficial to the reconstruction of the structural information of the image. FFDNet can retain more details, and whose PSNR is also higher than that of other methods, which verifies the advantages of the used FFDNet in improving the image restoration performance.

3) Ablation Experiments: To illustrate the effectiveness of the different terms of the proposed method, we discuss the contributions of low-rank quaternion prior and FFDNet for color image inpainting with the MR $= 0.85$ in Fig. 9. It can be seen that the matrix-based LRMC [48] and the Tucker-based HaLRTC [24] are slightly inferior to the quaternion-based LRQA [5]. Besides, compared with LRQA, the inpainting performance combined with the plug-and-play denoiser is significantly improved. The proposed DLRQP outperforms BM3D+LRQA and CBM3D+LRQA and is closer to the ground truth. As a whole, we can conclude that the unified framework of quaternion combined with deep denoiser prior is effective, and the FFDNet we utilized in the framework can achieve promising performance.
tends to converge. The reason for the fluctuations may be that the method has some fluctuations in the curve at the beginning of iterations increasing. During the period, the proposed DLRQP model compared to other competing methods. The potential research direction is how to extend the proposed DLRQP into other color image tasks, such as color image deblurring, color image recognition, color image super-resolution and so on.

### TABLE IV
**AVERAGE RUNNING TIME (IN SECONDS) OF THE PROPOSED AND COMPETING METHODS**

<table>
<thead>
<tr>
<th>Color Image Denoising</th>
<th>LSCD</th>
<th>KQ/SVD</th>
<th>WNNM</th>
<th>LRQA</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0058</td>
<td>1.16</td>
<td>4989.63</td>
<td>171.62</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Color Image Inpainting</th>
<th>LMaFit</th>
<th>TNR</th>
<th>WNNM</th>
<th>MC-NC</th>
<th>D-N</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.32</td>
<td>105.12</td>
<td>320.16</td>
<td>7.72</td>
<td>7.12</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 10: The convergence curves about relative error and iterations on Img1-Img12.

4) **Running Time:** We further report the average running time of all approaches on all images for color image denoising and inpainting as shown in Table IV. For color image inpainting, LMaFit has the shortest running time since it is based on matrix factorization without computationally time-consuming SVD. For color image denoising, the shortest time-consuming method is FFDNet, because FFDNet has been pre-trained and uses GPU-accelerated operations. Overall, the runtime of the proposed is comparable.

5) **Convergence Behaviors:** Fig. 10 illustrates the numerical convergence of our algorithm under MR = 0.85 on all images. It can be seen that the relative error decreases as the number of iterations increasing. During the period, the proposed DLRQP model has some fluctuations in the curve at the beginning of the iteration. The reason for the fluctuations may be that the deep denoiser incorporated in the proposed framework makes the solution of model unstable. Ultimately, the model still tends to converge.

### V. CONCLUSION

In this paper, we exploited the complementary advantages of deep plug-and-play prior and quaternion representation and proposed a novel color image processing model, termed as plug-and-play prior regularized low-rank quaternion representation. Unlike existing matrix-based and tensor-based methods, our DLRQP model was not only able to process three channels simultaneously and holistically by the quaternion representation, but also preserved the structural features of color images by pursuing the deep prior. Under the leverage of plugged deep denoiser, more image details could be preserved. Then, we applied the proposed DLRQP model to solve two color image processing applications, i.e., color image denoising and color image inpainting, and proposed optimization procedures, respectively. The results of numerical experiments illustrated the superiority of the proposed DLRQP model compared to other competing methods. The potential research direction is how to extend the proposed DLRQP into other color image tasks, such as color image deblurring, color image recognition, color image super-resolution and so on.

### REFERENCES


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